1.1 General Formulas

	$f(x)$	$g(y;\nu) = \int_0^\infty f(x)(xy)^{\frac12}J_\nu(xy)\,dx$
1.1	$\int_0^\infty g(y)(xy)^{\frac12}J_\nu(xy)\,dy$	$g(y)$
1.2	$f(ax), \quad a > 0$	$a^{-1}g(ya^{-1};\nu)$
1.3	$x^m f(x), m=0,1,2,\cdots$	$y^{\frac12-\nu}\left(\frac{d}{ydy}\right)^m [y^{m+\nu-\frac12}\, g(y;\nu+m)]$
1.4	$x^m f(x), m=0,1,2,\cdots$	$(-1)^m y^{\frac12+\nu}\left(\frac{d}{ydy}\right)^m [y^{m-\nu-\frac12}\, g(y;\nu-m)]$
1.5	$2\nu x^{-1} f(x)$	$yg(y;\nu-1)+yg(y;\nu+1)$
1.6	$x^{-\mu} f(x)$ $\mathrm{Re}\,\mu>0, \mathrm{Re}(\nu+1)>\mathrm{Re}\,\mu$	$2^{1-\mu}[\Gamma(\mu)]^{-1}y^{\frac12-\nu}\,\cdot$ $\cdot \int_0^y \tau^{\nu-\mu+\frac12}(y^2-\tau^2)^{\mu-1}g(\tau;\nu-\mu)\,d\tau$
1.7	$x^{-\mu} f(x)$ $\mathrm{Re}\,\mu > 0, \mathrm{Re}\,\nu-\frac32>\mathrm{Re}\,\mu$	$2^{1-\mu}[\Gamma(\mu)]^{-1}y^{\frac12+\nu}\,\cdot$ $\cdot \int_y^\infty \tau^{-\nu-\mu+\frac12}(\tau^2-y^2)^{\mu-1}g(\tau;\nu+\mu)\,d\tau$
1.8	$f'(x)$	$\frac12\nu^{-1}[(\nu-\frac12)yg(y;\nu+1)-(\nu+\frac12)yg(y;\nu-1)]$

	$f(x)$	$g(y) = \int_0^\infty f(x)(xy)^{\frac{1}{2}} J_0(xy)\,dx$
2.17	$x^{\frac{1}{2}}(x^4+2a^2x^2+b^4)^{-\frac{1}{2}}$ $b > a$	$y^{\frac{1}{2}} J_0[2^{-\frac{1}{2}} y(b^2-a^2)^{\frac{1}{2}}] \cdot$ $\cdot\, K_0[2^{-\frac{1}{2}} y(b^2+a^2)^{\frac{1}{2}}]$
2.18	$x^{\frac{1}{2}}(1+x^4)^{-\frac{1}{2}} \cdot$ $\cdot\, [x^2+(1+x^4)^{\frac{1}{2}}]^{-\nu}$ $\mathrm{Re}\,\nu > -3/4$	$y^{\frac{1}{2}} J_\nu(2^{-\frac{1}{2}} y) K_\nu(2^{-\frac{1}{2}} y)$
2.19	$x^{-\frac{1}{2}} e^{-ax}$	$y^{\frac{1}{2}}(a^2+y^2)^{-\frac{1}{2}}$
2.20	$x^{\frac{1}{2}}\, e^{-ax}$	$a y^{\frac{1}{2}}(a^2+y^2)^{-3/2}$
2.21	$x^{-3/2}(1-e^{-ax})$	$y^{\frac{1}{2}} \sinh(ay^{-1})$
2.22	$x^{-\frac{1}{2}} e^{-ax^2}$	$\frac{1}{2}\left(\frac{\pi}{a}\right)^{\frac{1}{2}} y^{\frac{1}{2}}\, e^{-\frac{y^2}{8a}}\, I_0\left(\frac{y^2}{8a}\right)$
2.23	$x^{\frac{1}{2}}\, e^{-ax^2}$	$(2a)^{-1}\, y^{\frac{1}{2}}\, e^{-\frac{y^2}{4a}}$
2.24	$x^{-3/2}\, e^{-\frac{a}{x}}$	$2y^{\frac{1}{2}} K_0[(2ay)^{\frac{1}{2}}] J_0[2ay)^{\frac{1}{2}}]$
2.25	$x^{-1}\, e^{-ax^{\frac{1}{2}}}$	$\frac{1}{8}\, \pi a y^{-\frac{1}{2}} \{[J_{\frac{1}{4}}(\frac{a^2}{8y})]^2 + [Y_{\frac{1}{4}}(\frac{a^2}{8y})]^2\}$

	$f(x)$	$g(y) = \int_0^\infty f(x)(xy)^{\frac{1}{2}} J_0(xy)\,dx$
2.26	$x^{\frac{1}{2}}\exp[-a(b^2+x^2)^{\frac{1}{2}}]$	$ay^{\frac{1}{2}}(a^2+y^2)^{-3/2}[1+b(a^2+y^2)^{\frac{1}{2}}] \cdot \exp[-b(a^2+y^2)^{\frac{1}{2}}]$
2.27	$x^{\frac{1}{2}}(b^2+x^2)^{-\frac{1}{2}} \cdot \exp[-a(b^2+x^2)^{\frac{1}{2}}]$	$y^{\frac{1}{2}}(a^2+y^2)^{-\frac{1}{2}} \exp[-b(a^2+y^2)^{\frac{1}{2}}]$
2.28	$x^{-\frac{1}{2}}\log x$	$-\, y^{-\frac{1}{2}} \log(2\gamma y)$
2.29	$x^{-\frac{1}{2}}(a^2+x^2)^{-\frac{1}{2}} \cdot \log[x+(a^2+x^2)^{\frac{1}{2}}]$	$y^{\frac{1}{2}}[\tfrac{1}{2}K_0^2(\tfrac{1}{2}ay) + \log a\; I_0(\tfrac{1}{2}ay)K_0(\tfrac{1}{2}ay)]$
2.30	$x^{-\frac{1}{2}}(a^2+x^2)^{-\frac{1}{2}} \cdot \log\left[\frac{(a^2+x^2)^{\frac{1}{2}}+x}{(a^2+x^2)^{\frac{1}{2}}-x}\right]$	$y^{\frac{1}{2}} K_0^2(\tfrac{1}{2}ay)$
2.31	$x^{\frac{1}{2}}(z^2-1)^{-\frac{1}{2}} \cdot \log[z+(z^2-1)^{\frac{1}{2}}]$ $z = (2ab)^{-1}(a^2+b^2+y^2)$	$2aby^{\frac{1}{2}} K_0(ay)\, K_0(by)$
2.32	$x^{\frac{1}{2}}(a^4+x^4)^{-\frac{1}{2}} \cdot \log\left[\frac{x^2+(a^4+x^4)^{\frac{1}{2}}}{a^2}\right]$	$-\, \tfrac{1}{2}\pi y^{\frac{1}{2}}\, Y_0(2^{-\frac{1}{2}}ay)\, K_0(2^{-\frac{1}{2}}ay)$

.5 Exponential and Logarithmic Functions

	$f(x)$	$g(y) = \int_0^\infty f(x)(xy)^{\frac{1}{2}} J_\nu(xy)\,dx$
5.1	$x^{-\frac{1}{2}} e^{-\alpha x}$ $\mathrm{Re}\ \nu > -1,\ \mathrm{Re}\ \alpha > 0$	$y^{\frac{1}{2}+\nu}(\alpha^2+y^2)^{-\frac{1}{2}}[\alpha+(\alpha^2+y^2)^{\frac{1}{2}}]^{-\nu}$
5.2	$x^{-3/2} e^{-\alpha x}$ $\mathrm{Re}\ \nu > 0,\ \mathrm{Re}\ \alpha > 0$	$\nu^{-1} y^{\frac{1}{2}+\nu}[\alpha+(\alpha^2+y^2)^{\frac{1}{2}}]^{-\nu}$
5.3	$x^{m+\frac{1}{2}} e^{-\alpha x}$ $\mathrm{Re}\ \nu > -m-2,\ \mathrm{Re}\ \alpha > 0$ $m = 0, 1, 2, \cdots$	$(-1)^{m+1} y^{\frac{1}{2}+\nu} \frac{d^{m+1}}{d\alpha^{m+1}} \{(\alpha^2+y^2)^{-\frac{1}{2}} \cdot$ $\cdot\ [\alpha+(\alpha^2+y^2)^{\frac{1}{2}}]^{-\nu}\}$
5.4	$x^{\nu+\frac{1}{2}} e^{-\alpha x}$ $\mathrm{Re}\ \nu > -1,\ \mathrm{Re}\ \alpha > 0$	$2^{\nu+1}\pi^{-\frac{1}{2}}\Gamma(\nu+3/2)\,\alpha y^{\frac{1}{2}+\nu}(\alpha^2+y^2)^{-\nu-3/2}$
5.5	$x^{\nu-\frac{1}{2}} e^{-\alpha x}$ $\mathrm{Re}\ \nu > -\frac{1}{2},\ \mathrm{Re}\ \alpha > 0$	$2^{\nu}\pi^{-\frac{1}{2}}\Gamma(\frac{1}{2}+\nu)\, y^{\nu+\frac{1}{2}}(\alpha^2+y^2)^{-\nu-\frac{1}{2}}$
5.6	$x^{\mu-\frac{1}{2}} e^{-\alpha x}$ $\mathrm{Re}(\mu+\nu) > -1,\ \mathrm{Re}\ \alpha > 0$	$y^{\frac{1}{2}}(\alpha^2+y^2)^{-\frac{1}{2}\mu-\frac{1}{2}}\Gamma(\nu+\mu+1)\,P_\mu^{-\nu}[\alpha(\alpha^2+y^2)^{-\frac{1}{2}}]$
5.7	$x^{-\frac{1}{2}} e^{-\alpha x^2}$ $\mathrm{Re}\ \nu > -1,\ \mathrm{Re}\ \alpha > 0$	$\frac{1}{2}(\pi \frac{y}{\alpha})^{\frac{1}{2}} \exp(-\frac{y^2}{8\alpha})\, I_{\frac{1}{2}\nu}(\frac{y^2}{8\alpha})$

	$f(x)$	$g(y) = \int_0^\infty f(x)(xy)^{\frac12} J_\nu(xy)\,dx$
5.8	$x^{\frac12} e^{-\alpha x^2}$ Re $\nu>-2$, Re $\alpha>0$	$\frac{1}{8}\pi^{\frac12}\alpha^{-3/2} y^{3/2} \exp(-\frac{y^2}{8\alpha}) \cdot$ $\cdot\ [I_{\frac12\nu-\frac12}(\frac{y^2}{8\alpha}) - I_{\frac12\nu+\frac12}(\frac{y^2}{8\alpha})]$
5.9	$x^{\nu+\frac12} e^{-\alpha x^2}$ Re $\nu>-1$, Re $\alpha>0$	$(2\alpha)^{-\nu-1}\, y^{\nu+\frac12} \exp(-\frac{y^2}{4\alpha})$
5.10	$x^{\nu-3/2} e^{-\alpha x^2}$ Re $\nu>0$, Re $\alpha>0$	$2^{\nu-1}\, y^{\frac12-\nu}\, \gamma(\nu, \frac{y^2}{4\alpha})$
5.11	$x^{\nu+\frac12}\, e^{\pm i\alpha x^2}$ $-1<$ Re $\nu<\frac12$, $\alpha>0$	$(2\alpha)^{-\nu-1}\, y^{\nu+\frac12} \exp[\pm i(\frac{\nu+1}{2}\pi - \frac{y^2}{4\alpha})]$
5.12	$x^{2n+\nu+\frac12}\, e^{-\frac14 x^2}$ Re $\nu>-1-2n, n=0,1,2,\ldots$	$2^{2n+\nu+1}\, n!\, y^{\nu+\frac12}\, e^{-y^2}\, L_n^{\nu}(y^2)$
5.13	$x^{\mu-\frac12}\, e^{-\alpha x^2}$ Re$(\mu+\nu)>-1$, Re $\alpha>0$	$\frac{\Gamma(\frac12+\frac12\nu+\frac12\mu)}{\Gamma(1+\nu)}\alpha^{-\frac12\mu} y^{-\frac12} \exp(-\frac{y^2}{8\alpha}) \cdot$ $\cdot\ M_{\frac12\mu,\frac12\nu}(\frac{y^2}{4\alpha})$
5.14	$x^{-3/2} e^{-\frac{\alpha}{x}}$, Re $\alpha>0$	$2y^{\frac12} J_\nu[(2\alpha y)^{\frac12}]\, K_\nu[(2\alpha y)^{\frac12}]$

	$f(x)$	$g(y) = \int_0^\infty f(x)(xy)^{\frac{1}{2}} J_\nu(xy)\,dx$
5.15	$x^{-3/2} e^{-\frac{\alpha}{x}-\beta x}$ Re $\alpha>0$, Re $\beta>0$	$2y^{\frac{1}{2}} J_\nu\{(2\alpha)^{\frac{1}{2}}[(\beta^2+y^2)^{\frac{1}{2}}-\beta]^{\frac{1}{2}}\}\cdot$ $\cdot K_\nu\{(2\alpha)^{\frac{1}{2}}[(\beta^2+y^2)^{\frac{1}{2}}+\beta]^{\frac{1}{2}}\}$
5.16	$x^{-1} e^{-(\alpha x)^{\frac{1}{2}}}$ Re $\nu>-\frac{1}{2}$, Re $\alpha>0$	$\pi^{-\frac{1}{2}} 2^{\frac{1}{2}} \Gamma(\frac{1}{2}+\nu) D_{-\nu-\frac{1}{2}}[e^{i\frac{\pi}{4}}(\frac{2y}{\alpha})^{-\frac{1}{2}}] D_{-\nu-\frac{1}{2}}[e^{-i\frac{\pi}{4}}(2\frac{y}{a})^{-\frac{1}{2}}]$
5.17	$x^{\nu+\frac{1}{2}} e^{\alpha(1-x^2)}$, $x<1$ 0, $x>1$ Re $\nu>-\frac{1}{2}$	$(2i\alpha)^{-\nu-1} y^{\frac{1}{2}+\nu}[U_{\nu+1}(2i\alpha,y)-iU_{\nu+2}(2i\alpha,y)]$
5.18	$x^{\nu+\frac{1}{2}} \exp[-\alpha(b^2+x^2)^{\frac{1}{2}}]$ Re $\nu>-1$, Re $\alpha>0$	$(\frac{1}{2}\pi)^{-\frac{1}{2}} \alpha b^{\nu+3/2} y^{\frac{1}{2}+\nu}(y^2+\alpha^2)^{-\nu-3/4}\cdot$ $\cdot K_{\nu+3/2}[b(y^2+\alpha^2)^{\frac{1}{2}}]$
5.19	$x^{-\frac{1}{2}}(b^2+x^2)^{-\frac{1}{2}}\cdot$ $\cdot\exp[-\alpha(b^2+x^2)^{\frac{1}{2}}]$ Re $\nu>-1$, Re $\alpha>0$	$y^{\frac{1}{2}} I_{\frac{1}{2}\nu}\{\frac{1}{2}b[(\alpha^2+y^2)^{\frac{1}{2}}-\alpha] K_{\frac{1}{2}\nu}\{\frac{1}{2}b[(\alpha^2+y^2)^{\frac{1}{2}}+\alpha]$
5.20	$x^{\nu+\frac{1}{2}}(b^2+x^2)^{-\frac{1}{2}}\cdot$ $\cdot\exp[-\alpha(b^2+x^2)^{\frac{1}{2}}]$ Re $\nu>-1$, Re $\alpha>0$	$(\frac{1}{2}\pi)^{-\frac{1}{2}} b^{\nu+\frac{1}{2}} y^{\nu+\frac{1}{2}}(\alpha^2+y^2)^{-\frac{1}{2}\nu-\frac{1}{4}}\cdot$ $\cdot K_{\nu+\frac{1}{2}}[b(\alpha^2+y^2)^{\frac{1}{2}}]$

	$f(x)$	$g(y) = \int_0^\infty f(x)(xy)^{\frac{1}{2}} J_\nu(xy)\,dx$
5.21	$x^{\frac{1}{2}-\nu}(b^2+x^2)^{-\frac{1}{2}} \cdot$ $\cdot[(x^2+b^2)^{\frac{1}{2}}-b]^\nu \cdot$ $\cdot\exp[-\alpha(x^2+b^2)^{\frac{1}{2}}]$ $\mathrm{Re}\ \nu>-1,\ \mathrm{Re}\ \alpha>0$	$y^{\nu+\frac{1}{2}}[\alpha+(\alpha^2+y^2)^{\frac{1}{2}}]^{-\nu}(\alpha^2+y^2)^{-\frac{1}{2}} \cdot$ $\cdot \exp[-b(\alpha^2+y^2)^{\frac{1}{2}}]$
5.22	$x^{\tau-\frac{1}{2}}(b^2+x^2)^{-\frac{1}{2}} \cdot$ $\cdot\ [(b^2+x^2)^{\frac{1}{2}}+b]^{-\tau} \cdot$ $\cdot\ \exp[-\alpha(b^2+x^2)^{\frac{1}{2}}]$ $\mathrm{Re}(\nu+\tau) > -1$ $\mathrm{Re}\ \alpha > 0$	$\frac{\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\tau)}{b\Gamma(1+\nu)}\ y^{-\frac{1}{2}} M_{\frac{1}{2}\tau,\frac{1}{2}\nu}\{b[(\alpha^2+y^2)^{\frac{1}{2}}-\alpha]\} \cdot$ $\cdot\ W_{-\frac{1}{2}\tau,\frac{1}{2}\nu}\{b[(\alpha^2+y^2)^{\frac{1}{2}}+\alpha]\}$

Appendix. List of Notations and Definitions

Abbreviations: ε_n = Neumann's number

$\varepsilon_0 = 1, \quad \varepsilon_n = 2, \quad n = 1, 2, 3,$

$\gamma = 0.57721...$ Euler's constant

1. Elementary functions

Trigonometric and inverse trigonometric functions:

$$\sin x, \quad \cos x, \quad \tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}$$

$$\sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}, \quad \arcsin x, \quad \arccos x,$$

$$\arctan x, \quad \operatorname{arccot} x.$$

Hyperbolic functions:

$$\sinh x = \tfrac{1}{2}(e^{x}-e^{x}), \quad \cosh x = \tfrac{1}{2}(e^{x}+e^{-x})$$

$$\tanh x = \frac{\sinh x}{\cosh x}, \quad \coth x = \frac{\cosh x}{\sinh x},$$

$$\operatorname{sech} x = \frac{1}{\cosh x}, \quad \operatorname{csch} x = \frac{1}{\sinh x}.$$

2. Orthogonal polynomials

Legendre polynomials $P_n(x)$.

$$P_n(x) = 2^{-n}(n!)^{-1}\frac{d^n}{dx^n}(x^2-1)^n = {}_2F_1(-n, n+1; 1; \tfrac{1}{2}-\tfrac{1}{2}x)$$

Gegenbauer's polynomials $C_n^{\nu}(x)$

$$C_n^{\nu}(x) = [n!\Gamma(2\nu)]^{-1}\Gamma(2\nu+n)\,{}_2F_1(-n, 2\nu+n; \tfrac{1}{2}+\nu; \tfrac{1}{2}-\tfrac{1}{2}x)$$

Chebycheff polynomials $T_n(x)$, $U_n(x)$

$$T_n(x) = \cos(n \arccos x) = {}_2F_1(-n,n;\tfrac{1}{2};\tfrac{1}{2}-\tfrac{1}{2}x) = \tfrac{1}{2}n \lim_{\nu=0} \cdot \Gamma(\nu) C_n^{\nu}(x)$$

$$= \tfrac{1}{2}\{[x+i(1-x^2)^{\frac{1}{2}}]^n + [x-i(1-x^2)^{\frac{1}{2}}]^n\}$$

$$U_n(x) = (1-x^2)^{-\frac{1}{2}} \sin[(n+1)\arccos x] = C_n^1(x)$$

$$= x(n+1)\, {}_2F_1(\tfrac{1}{2}-\tfrac{1}{2}n, \tfrac{3}{2}+\tfrac{1}{2}n; \tfrac{3}{2}; 1-x^2)$$

$$= -\tfrac{1}{2}i(1-x^2)^{-\frac{1}{2}}\{[x+i(1-x^2)^{\frac{1}{2}}]^n - [x-i(1-x^2)^{\frac{1}{2}}]^n\}$$

Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$

$$P_n^{(\alpha,\beta)}(x) = [n!\Gamma(1+\alpha)]^{-1}\Gamma(1+\alpha+n)\, {}_2F_1(-n,n+\alpha+\beta+1;\alpha+1;\tfrac{1}{2}-\tfrac{1}{2}x)$$

Laguerre polynomials

$$L_n^{\alpha}(x) = (n!)^{-1} x^{-\alpha} e^{x} \frac{d^n}{dx^n} (e^{-x} x^{n+\alpha})$$

$$= [n!\Gamma(1+\alpha)]^{-1}\Gamma(\alpha+1+n)\, {}_1F_1(-n;\alpha+1;x)$$

$$L_n^0(x) = L_n(x)$$

Hermite polynomials

$$He_n(x) = (-1)^n \exp(\tfrac{1}{2}x^2) \frac{d^n}{dx^n} \exp(-\tfrac{1}{2}x^2)$$

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} = 2^{\frac{1}{2}n} He_n(2^{\frac{1}{2}}x)$$

$$He_{2n}(x) = (-1)^n 2^{-n} (n!)^{-1} (2n)!\, {}_1F_1(-n;\tfrac{1}{2};\tfrac{1}{2}x^2)$$

$$He_{2n+1}(x) = x(-1)^n 2^{-n}(n!)^{-1}(2n+1)!\,{}_1F_1(-n;\tfrac{3}{2};\tfrac{1}{2}x^2)$$

3. The Gamma function and related functions

$$\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1}\, dt \quad , \qquad \text{Re } z > 0$$

ψ-function

$$\psi(z) = \frac{d}{dz} \log z = \frac{\Gamma'(z)}{\Gamma(z)}$$

Beta function $B(x,y)$

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

4. Legendre functions

(Definition according to Hobson)

$$P_\nu^\mu(z) = [\Gamma(1-\mu)]^{-1}\left(\frac{z+1}{z-1}\right)^{\frac{1}{2}\mu} {}_2F_1(-\nu,\nu+1;1-\mu;\tfrac{1}{2}-\tfrac{1}{2}z)$$

$$e^{-i\pi\mu} q_\nu^\mu(z) = 2^{-\nu-1}[\Gamma(\tfrac{3}{2}+\nu)]^{-1}\pi^{\frac{1}{2}}\Gamma(1+\nu+\mu)\, z^{-\nu-\mu-1}(z^2-1)^{\frac{1}{2}\mu} \cdot$$

$$\cdot\ {}_2F_1(\tfrac{1}{2}\nu+\tfrac{1}{2}\mu+\tfrac{1}{2}, \tfrac{1}{2}\nu+\tfrac{1}{2}\mu+1; \nu+\tfrac{3}{2}; z^{-2})$$

z is a point in the complex z-plane cut along the real z-axis from $-\infty$ to $+1$

$$(z^2-1)^{\frac{1}{2}\mu} = (z-1)^{\frac{1}{2}\mu}(z+1)^{\frac{1}{2}} \ , \ -\pi < \arg z < \pi, \ -\pi < \arg(z\pm 1) < \pi$$

$$P_\nu^\mu(x) = [\Gamma(1-\mu)]^{-1}\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}\mu} {}_2F_1(-\nu,\nu+1;1-\mu;\tfrac{1}{2}-\tfrac{1}{2}x), \ -1 < x < 1$$

$$Q_\nu^\mu(x) = \tfrac{1}{2}e^{-i\pi\mu}[e^{-\frac{1}{2}i\pi\mu} q_\nu^\mu(x+i0) + e^{\frac{1}{2}i\pi\mu} q_\nu^\mu(x-i0)], \ -1 < x < 1$$

$$p_\nu^\circ(z) = p_\nu(z); \qquad q_\nu^\circ(z) = q_\nu(z)$$

$$P_\nu^\circ(z) = P_\nu(z); \qquad Q_\nu^\circ(z) = Q_\nu(z)$$

5. <u>Bessel functions</u>

$$J_\nu(z) = (\tfrac{1}{2}z)^\nu \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{z}{2})^{2n}}{n!\Gamma(\nu+n+1)}$$

$$J_{-\nu}(z) = J_\nu(z)\cos(\pi\nu) - Y_\nu(z)\sin(\pi\nu);$$

$$Y_{-\nu}(z) = J_\nu(z)\sin(\pi\nu) + Y_\nu(z)\cos(\pi\nu);$$

$$Y_\nu(z) = \cot(\pi\nu)J_\nu(z) - \csc(\pi\nu)J_{-\nu}(z)$$

$$H_\nu^{(1)}(z) = J_\nu(z)+iY_\nu(z); \quad H_\nu^{(2)}(z) = J_\nu(z) - iY_\nu(z)$$

6. <u>Modified Bessel functions</u>

$$I_\nu(z) = e^{-\frac{1}{2}i\pi\nu}J_\nu(ze^{i\frac{1}{2}\pi}) = (\tfrac{1}{2}z)^\nu \sum_{n=0}^{\infty} \frac{(\frac{1}{2}z)^{2n}}{n!\Gamma(\nu+n+1)}$$

$$K_\nu(z) = \tfrac{1}{2}\pi \csc(\pi\nu) \; [I_{-\nu}(z) - I_\nu(z)]$$

$$= \tfrac{1}{2}i\pi e^{\frac{1}{2}i\pi\nu}H_\nu^{(1)}(ze^{i\frac{1}{2}\pi}) = -\tfrac{1}{2}i\pi e^{-i\frac{1}{2}\pi\nu}H_\nu^{(2)}(ze^{-i\frac{1}{2}\pi\nu})$$

7. <u>Anger-Weber functions</u>

$$\mathbf{J}_\nu(z) = \pi^{-1}\int_0^\pi \cos(z\,\mathrm{sin}t-\nu t)\,dt$$

$$\mathbf{E}_\nu(z) = -\pi^{-1}\int_0^\pi \sin(z\,\mathrm{sin}t-\nu t)\,dt$$

$$\mathbf{J}_n(z) = J_n(z), \qquad n = 0, 1, 2, \cdots ; \qquad \mathbf{E}_0(z) = -\mathbf{H}_0(z)$$

$$\mathbf{J}_{\frac{1}{2}}(z) = (\tfrac{1}{2}\pi z)^{-\frac{1}{2}}\{[C(z)-S(z)]\cos z+[C(z)+S(z)]\sin z\} = \mathbf{E}_{-\frac{1}{2}}(z)$$

$$\mathbf{J}_{-\frac{1}{2}}(z) = (\tfrac{1}{2}\pi z)^{-\frac{1}{2}}\ [C(z)+S(z)]\cos z-[C(z)-S(z)]\sin z\} = \mathbf{E}_{\frac{1}{2}}(z)$$

8. Struve functions

$$\mathbf{H}_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n(\frac{1}{2}z)^{\nu+2n+1}}{\Gamma(n+\frac{3}{2})\Gamma(\nu+n+\frac{3}{2})}$$

$$\mathbf{L}_\nu(z) = -ie^{-\frac{1}{2}i\pi\nu}\mathbf{H}_\nu(ze^{i\frac{1}{2}\pi}) = \sum_{n=0}^{\infty} \frac{(\frac{1}{2}z)^{\nu+2n+1}}{\Gamma(n+\frac{3}{2})\Gamma(\nu+n+\frac{3}{2})}$$

9. Lommel functions

$$s_{\mu,\nu}(z) = \frac{z^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)}\ {}_1F_2(1;\tfrac{1}{2}\mu-\tfrac{1}{2}\nu+\tfrac{3}{2},\tfrac{1}{2}\mu+\tfrac{1}{2}\nu+\tfrac{3}{2};-\tfrac{1}{4}z^2)$$

$$\mu\pm\nu \neq -1, -2, -3, \cdots$$

$$S_{\mu,\nu}(z) = s_{\mu,\nu}(z)-2^{\mu-1}\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\mu-\tfrac{1}{2}\nu)\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\mu+\tfrac{1}{2}\nu)\ \cdot$$

$$\cdot\ \{\sin[\tfrac{1}{2}\pi(\nu-\mu)]J_\nu(z)+\cos[\tfrac{1}{2}\pi(\nu-\mu)]Y_\nu(z)\}$$

$$s_{\nu,\mu}(z) = s_{\nu,-\mu}(z); \qquad S_{\mu,\nu}(z) = S_{\mu,-\nu}(z)$$

Special cases:

$$s_{\nu,\nu}(z) = \Pi^{\frac{1}{2}}2^{\nu-1}\Gamma(\tfrac{1}{2}+\nu)\mathbf{H}_\nu(z)$$

$$S_{\nu,\nu}(z) = \pi^{\frac{1}{2}}2^{\nu-1}\Gamma(\tfrac{1}{2}+\nu)[\mathbf{H}_\nu(z) - Y_\nu(z)]$$

$$s_{\nu+1,\nu}(z) = z^{\nu}-2^{\nu}\Gamma(1+\nu)J_{\nu}(z); \qquad S_{\nu+1,\nu} = z^{\nu}$$

$$\lim_{\mu\to\nu} \frac{S_{\mu-1,\nu}(z)}{\Gamma(\mu-\nu)} = 2^{\nu-1}\Gamma(\nu)J_{\nu}(z)$$

$$s_{0,\nu}(z) = \tfrac{1}{2}\pi\csc(\pi\nu)\,[\mathbf{J}_{\nu}(z) - \mathbf{J}_{-\nu}(z)]$$

$$S_{0,\nu}(z) = \tfrac{1}{2}\pi\csc(\pi\nu)\,[\mathbf{J}_{\nu}(z) - \mathbf{J}_{-\nu}(z) - J_{\nu}(z) + J_{-\nu}(z)]$$

$$s_{-1,\nu}(z) = -\tfrac{1}{2}\pi\nu^{-1}\csc(\pi\nu)\,[\mathbf{J}_{\nu}(z)+\mathbf{J}_{-\nu}(z)]$$

$$S_{-1,\nu}(z) = \tfrac{1}{2}\pi\nu^{-1}\csc(\pi\nu)\,[J_{\nu}(z)+J_{-\nu}(z)-\mathbf{J}_{\nu}(z)-\mathbf{J}_{-\nu}(z)]$$

$$s_{1,\nu}(z) = 1-\tfrac{1}{2}\pi\nu\csc(\pi\nu)\,[\mathbf{J}_{\nu}(z)+\mathbf{J}_{-\nu}(z)]$$

$$S_{1,\nu}(z) = 1+\tfrac{1}{2}\pi\nu\csc(\pi\nu)\,[J_{\nu}(z)+J_{-\nu}(z)-\mathbf{J}_{\nu}(z)-\mathbf{J}_{-\nu}(z)]$$

$$S_{\frac{1}{2},\frac{1}{2}}(z) = z^{-\frac{1}{2}}, \qquad S_{\frac{3}{2},\frac{1}{2}}(z) = z^{\frac{1}{2}}$$

$$S_{-\frac{1}{2},\frac{1}{2}}(z) = z^{-\frac{1}{2}}[\sin z\ \mathrm{Ci}(z)-\cos z\ \mathrm{si}(z)]$$

$$S_{-\frac{3}{2},\frac{1}{2}}(z) = -z^{-\frac{1}{2}}[\sin z\ \mathrm{si}(z)+\cos z\ \mathrm{Ci}(z)]$$

Lommel functions of two variables

$$U_{\nu}(w,z) = \sum_{n=0}^{\infty} (-1)^{n}\left(\frac{w}{z}\right)^{\nu+2n} J_{\nu+2n}(z)$$

$$V_{\nu}(w,z) = \cos(\tfrac{1}{2}w+\tfrac{1}{2}z^{2}w^{-1}+\tfrac{1}{2}\pi\nu)+U_{2-\nu}(w,z)$$

Kelvin's functions

$$J_{\nu}(ze^{i\frac{3}{4}\pi}) = \mathrm{ber}_{\nu}(z)+i\,\mathrm{bei}_{\nu}(z)$$

$$J_{\nu}(ze^{-i\frac{3}{4}\pi}) = \mathrm{ber}_{\nu}(z)-i\,\mathrm{bei}_{\nu}(z)$$

$$K_\nu(ze^{i\frac{\pi}{4}}) = \mathrm{ker}_\nu(z)+i\ \mathrm{kei}_\nu(z)$$

$$K_\nu(ze^{-i\frac{\pi}{4}}) = \mathrm{ker}_\nu(z)-i\ \mathrm{kei}_\nu(z)$$

$$\mathrm{ber}_0(z) = \mathrm{ber}(z),\ \mathrm{bei}_0(z)=\mathrm{bei}(z),\ \mathrm{ker}_0(z)=\mathrm{ker}(z),$$
$$\mathrm{kei}_0(z)=\mathrm{kei}(z)$$

10. Gauss' hypergeometric series

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!}, \qquad |z| < 1$$

11. Confluent hypergeometric functions

Kummer's functions

$${}_1F_1(a;c;z) = \frac{\Gamma(c)}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(c+n)} \frac{z^n}{n!} = z^{-\frac{1}{2}c}\ e^{\frac{1}{2}z}\ M_{\frac{1}{2}c-a,\frac{1}{2}c-\frac{1}{2}}(z)$$

$${}_1F_1(a;c;z) = e^z\ {}_1F_1(c-a;c;-z)$$

Whittaker functions

$$M_{k,\mu}(z) = z^{\mu+\frac{1}{2}}e^{-\frac{1}{2}z}\ {}_1F_1(\tfrac{1}{2}+\mu-k;2\mu+1;z)$$
$$= z^{\mu+\frac{1}{2}}e^{\frac{1}{2}z}\ {}_1F_1(\tfrac{1}{2}+\mu+k;2\mu+1;-z)$$

$$W_{k,\mu}(z) = \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-k)} M_{k,\mu}(z) + \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu-k)} M_{k,-\mu}(z)$$

$$W_{k,-\mu}(z) = W_{k,\mu}(z)$$

Parabolic cylinder function

$$D_{\alpha}(z) = 2^{\frac{1}{4}+\frac{1}{2}\alpha} z^{-\frac{1}{2}} W_{\frac{1}{4}+\frac{1}{2}\alpha,\frac{1}{4}}(\tfrac{1}{2}z^2)$$

$$D_n(z) = e^{-\frac{1}{4}z^2} He_n(z) , \qquad n = 0, 1, 2, \cdots$$

$$D_{-1}(z) = (\tfrac{1}{2}\pi)^{\frac{1}{2}} e^{\frac{1}{4}z^2} \; Erfc(2^{-\frac{1}{2}}z)$$

$$D_{-\frac{1}{2}}(z) = (2\pi z^{-1})^{-\frac{1}{2}} K_{\frac{1}{4}}(\tfrac{1}{4}z^2)$$

Error integrals

$$Erf(z) = 2\pi^{-\frac{1}{2}} \int_0^z e^{-t^2} dt = 2\pi^{-\frac{1}{2}} z \, {}_1F_1(\tfrac{1}{2}; \tfrac{3}{2}; -z^2) = 2(\pi z)^{-\frac{1}{2}} e^{-\frac{1}{2}z^2} M_{-\frac{1}{4},\frac{1}{4}}(z^2)$$

$$Erfc(z) = 1-Erf(z) = 2\pi^{-\frac{1}{2}} \int_z^{\infty} e^{-t^2} dt = (\pi z)^{-\frac{1}{2}} e^{-\frac{1}{2}z^2} W_{-\frac{1}{4},\frac{1}{4}}(z^2)$$

$$Erf(x^{\frac{1}{2}} e^{i\frac{\pi}{4}}) = C(x) + S(x) + i[C(x) - S(x)]$$

$$Erfc(x^{\frac{1}{2}} e^{i\frac{\pi}{4}}) = 1-C(x)-S(x)+i[S(x)-C(x)]$$

Fresnel's integrals

$$C(x) = (2\pi)^{-\frac{1}{2}} \int_0^z t^{-\frac{1}{2}} \cos t \, dt; S(x) = (2\pi)^{-\frac{1}{2}} \int_0^z t^{-\frac{1}{2}} \sin t \, dt$$

Exponential integral

$$-Ei(-z) = \int_z^{\infty} t^{-1} e^{-t} dt = -\gamma - \log z - \sum_{u=1}^{\infty} \frac{(-z)^n}{n \cdot n!} = z^{-\frac{1}{2}} e^{-\frac{1}{2}z} W_{-\frac{1}{2},0}(z)$$

$$-\pi < \arg z < \pi$$

$$\overline{Ei}(x) = \tfrac{1}{2}[Ei(x+i0)+Ei(x-i0)] = -P.V. \int_{-x}^{\infty} t^{-1}e^{-t}dt, \; x > 0$$

(P.V. = principle value)

$$\overline{Ei}(x) = \gamma + \log x + \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!}, \qquad x > 0$$

$$Ei(-ix) = Ci(x) - isi(x); \qquad \bar{E}i(ix) = i\pi + Ci(x) + isi(x)$$

Sine and cosine integral

$$Si(x) = \int_0^x t^{-1} \sin t \, dt; \; si(x) = -\int_x^{\infty} t^{-1} \sin t \, dt = Si(x) - \frac{\pi}{2}$$

$$= \tfrac{1}{2}i[Ei(-ix) - Ei(ix)]$$

$$Ci(x) = -\int_x^{\infty} t^{-1} \cos t \, dt = \tfrac{1}{2}[Ei(-ix) + Ei(ix)]$$

$$- \gamma + \log x + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)(2n)!}$$

Incomplete gamma function

$$\gamma(\nu, z) = \int_0^z t^{\nu-1} e^{-t} dt = \nu^{-1} z^{\nu} \; {}_1F_1(\nu, \nu+1; -z), \; \text{Re } \nu > 0$$

$$= \nu^{-1} z^{\frac{1}{2}\nu - \frac{1}{2}} e^{-\frac{1}{2}z} M_{\frac{1}{2}\nu - \frac{1}{2}, \frac{1}{2}\nu}(z)$$

$$\Gamma(\nu, z) = \Gamma(\nu) - \gamma(\nu, z) = \int_z^{\infty} t^{\nu-1} e^{-t} dt$$

$$= z^{\frac{1}{2}\nu - \frac{1}{2}} e^{-\frac{1}{2}z} W_{\frac{1}{2}\nu - \frac{1}{2}, \frac{1}{2}\nu}(z)$$

$$\Gamma(\tfrac{1}{2}, z^2) = \pi^{\frac{1}{2}} \text{Erfc}(z); \qquad \Gamma(0, z) = -Ei(-z)$$

$$\gamma(\tfrac{1}{2}, z^2) = \pi^{\frac{1}{2}} \text{Erf}(z); \qquad \gamma(1, z) = 1 - e^{-z}$$

12. Generalized hypergeometric series

$${}_mF_n(a_1,a_2,\cdots,a_m;\ b_1,b_2,\cdots b_n;\ z) =$$

$$= \frac{\Gamma(b_1)\cdots\Gamma(b_n)}{\Gamma(a_1)\cdots\Gamma(a_m)} \sum_{k=0}^{\infty} \frac{\Gamma(a_1+k)\cdots\Gamma(a_m+k)}{\Gamma(b_1+k)\cdots\Gamma(b_n+k)} \frac{z^k}{k!}$$

For $m = n+1$, $|z| < 1$ For $m < n+1$, $|z| < \infty$

$m, n = 0, 1, 2 \cdots$

13. Elliptic integrals

Complete elliptic integrals

$$\mathbf{K}(k) = \int_0^{\frac{1}{2}\pi} (1-k^2\sin^2 x)^{-\frac{1}{2}} dx = \tfrac{1}{2}\pi\ {}_2F_1(\tfrac{1}{2},\tfrac{1}{2};1;k^2)$$

$$\mathbf{E}(k) = \int_0^{\frac{1}{2}\pi} (1-k^2\sin^2 x)^{\frac{1}{2}} dx = \tfrac{1}{2}\pi\ {}_2F_1(-\tfrac{1}{2},\tfrac{1}{2};1;k^2)$$

14. Particular cases of Whittaker's functions

$$M_{-\frac{1}{4},\frac{1}{4}}(z) = \tfrac{1}{2}\pi^{\frac{1}{2}} z^{\frac{1}{4}} e^{\frac{1}{2}z} \mathrm{Erf}(z^{\frac{1}{2}})$$

$$M_{\frac{1}{4},\frac{1}{4}}(z) = -i\tfrac{1}{2}\pi^{\frac{1}{2}} z^{\frac{1}{4}} e^{-\frac{1}{2}z} \mathrm{Erf}(iz^{\frac{1}{2}})$$

$$M_{k,0}(z) = z^{\frac{1}{2}} e^{-\frac{1}{2}z} L_{k-\frac{1}{2}}(z)$$

$$M_{0,\mu}(z) = 2^{2\mu}\Gamma(1+\mu) z^{\frac{1}{2}} I_\mu(\tfrac{1}{2}z)$$

$$M_{\mu+\frac{1}{2},\mu}(z) = z^{\mu+\frac{1}{2}} e^{-\frac{1}{2}z}$$

$$M_{-\mu-\frac{1}{2},\mu}(z) = z^{\mu+\frac{1}{2}}\, e^{\frac{1}{2}z}$$

$$M_{k,k+\frac{1}{2}}(z) = (2k+1)\, z^{-k}\, e^{\frac{1}{2}z}\, \gamma(2k+1,z)$$

$$W_{-\frac{1}{4},\frac{1}{4}}(z) = \pi^{\frac{1}{2}} z^{\frac{1}{4}} e^{\frac{1}{2}z}\, \mathrm{Erfc}(z^{\frac{1}{2}})$$

$$W_{k,\frac{1}{4}}(z) = 2^{\frac{1}{2}-k} z^{\frac{1}{4}}\, D_{2k-\frac{1}{2}}[(2z)^{\frac{1}{2}}]$$

$$W_{0,\mu}(z) = \pi^{-\frac{1}{2}} z^{\frac{1}{2}}\, K_{\mu}(\tfrac{1}{2}z)$$

$$W_{-\frac{1}{2},0}(z) = -z^{\frac{1}{2}} e^{\frac{1}{2}z}\, \mathrm{Ei}(-z)$$

$$W_{k,\frac{1}{2}+k}(z) = z^{-k} e^{\frac{1}{2}z}\, \Gamma(2k+1,z)$$

$$W_{k,k-\frac{1}{2}}(z) = z^{\mu+\frac{1}{2}} e^{-\frac{1}{2}z}$$

List of Functions

Symbol	Name of the function		Listed under
$C(x)$	Fresnel's integral		11
$Ci(x)$	Cosine integral		11
$C_n^{\nu}(x)$	Gegenbauer's polynomial		2
$D_{\nu}(z)$	Parabolic cylinder function		11
$\mathbf{E}(k)$	Complete elliptic integral		13
$Ei(-x)$, $\overline{Ei}(x)$	Exponential integrals		11
$Erf(z)$, $Erfc(z)$	Error integrals		11
$\mathbf{E}_{\nu}(z)$	Anger-Weber functions		7
${}_mF_n(z)$	Hypergeometric function		10, 11, 12
$He_n(x)$	Hermite's polynomial		2
$H_{\nu}^{(1),(2)}(z)$	Hankel's functions		5
$\mathbf{H}_{\nu}(z)$	Struve's function		8
$I_{\nu}(z)$	Modified Bessel function		6
$J_{\nu}(z)$	Bessel function		5
$\mathbf{J}_{\nu}(z)$	Anger-Weber function		7
$\mathbf{K}(k)$	Complete elliptic integral		13
$K_{\nu}(z)$	Modified Hankel function		6

Symbol	Name of the function	Listed under
$L_\nu(z)$	Laguerre's function	11
$L_n^\alpha(x)$	Laguerre's polynomial	2
$\mathbf{L}_\nu(z)$	Struve's functions	8
$M_{k,\mu}(z)$	Whittaker's function	11
$W_{k,\mu}(z)$		
$P_n(x)$	Legendre's polynomials	2
$P_n^{(\alpha,\beta)}(x)$	Jacobi's polynomials	2
$p_\nu^\mu(z)$	Legendre functions	4
$P_\nu^\mu(x)$		
$q_\nu^\mu(z)$		
$Q_\nu^\mu(x)$		
$S(x)$	Fresnel's integral	11
$si(x)$	Sine integrals	11
$Si(x)$		
$s_{\mu,\nu}(z)$	Lommel's functions	9
$S_{\mu,\nu}(z)$		