Prof. Mark D Shattuck
Physics 34100 Applied Differential Equations
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Problem Set 3

Question 1. *Exponential Function:*

(1) Find t:
  (a) $e^{at} = e$.
  (b) $e^{at} = e^3$.
  (c) $e^{a(t+2)} = e^t e^{2a}$.

(2) Solve $y' = ay$ when $y(T) = 1$.

(3) A dual number $z = x + y\epsilon$ is similar to a complex number, but $\epsilon^2 = 0$ instead of -1. x is the real part of z and y is the dual part of z.
  (a) Show that if $\epsilon = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then $\epsilon^2 = 0$.
  (b) The full matrix representation of a dual number $z = xI + y\epsilon = \begin{bmatrix} x & y \\ 0 & x \end{bmatrix}$, where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the 2×2 identity. Show that $I\epsilon = \epsilon I$.
  (c) Find $(x + \epsilon)^0$, $(x + \epsilon)^1$, $(x + \epsilon)^2$, and in general $(x + \epsilon)^N$. (Note: from above $x\epsilon = \epsilon x$.)
  (d) What is the relationship between the real part of $(x + \epsilon)^N$ and the dual part?
  (e) Find $\sin(x + \epsilon)$.

(4) There are many ways to convert a continuous differential equation $y' = ay$ into a discrete difference equation. The lhs could be

$$y' \simeq \frac{y(t + \Delta) - y(t)}{\Delta}$$

or

$$y' \simeq \frac{y(t) - y(t - \Delta)}{\Delta}$$

There is a similar ambiguity for the rhs $y(t)$ or $y(t + \Delta)$. From these there are two distinct possibilities for difference equations:

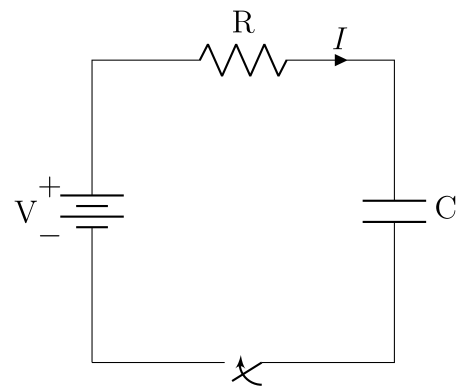
$$\begin{aligned} \frac{y(t) - y(t - \Delta)}{\Delta} &= ay(t - \Delta) \\ Y_n - Y_{n-1} &= a\Delta Y_{n-1}, \text{ where } t = n\Delta \text{ and } Y_n = y(t). \\ Y_n &= (1 + a\Delta)Y_{n-1} \\ Y_N &= (1 + a\Delta)^N Y_0 \end{aligned}$$

or

$$\begin{aligned} \frac{y(t) - y(t - \Delta)}{\Delta} &= ay(t) \\ Y_m - Y_{m-1} &= a\Delta Y_m. \end{aligned}$$

The first one is the ordinary discrete interest problem we discussed in class. As an interest problem it is exact, but we often use the continuous version to approximate. When assuming the continuous equation is exact then the first one is called the forward-Euler approximation. Forward refers to the fact that the rhs contains only already known values, and we can move forward directly. The second one is called backward-Euler. The change in Y depends on values we do not know yet.

(a) Solve the backward-Euler equation to find Y_m in terms of Y_{m-1}.
(b) Then solve for Y_m in terms of Y_0.
(c) As $\Delta \to 0$ or $N \to \infty$ both solutions converge to the continuous solution $y(t) = y(0)e^{at}$. Show that backward-Euler $\geq$ continuous $\geq$ forward-Euler by comparing $(1+\Delta)$, $\frac{1}{1-\Delta}$, and e^{Δ}. (Hint: Expand in Taylor-series).



Question 2. *Input-Response:*

(1) Solve the following equations for $y(t)$. If a steady-state exist, find it and write the solution $y(t) = y_p + y_N$ where $y_p = y(\infty) \equiv y_\infty$.
  (a) $y' = 5 - 2y$ given $y(0) = 10$.
  (b) $y' = 3 + 2y$ given $y(0) = 0$.
  (c) $y' = 2e^{3t} - 2y$ given $y(0) = 2$.
  (d) $y' = 2t^2 - 2y$ given $y(0) = 3$.
  (e) $y' = sin(3t) - 2y$ given $y(0) = 0$.
  (f) $y' = 1 - 2ty$ given $y(0) = 1$.

(2) In the circuit above, following the voltage around the loop when the switch is closed:

$$V - IR - \frac{q}{C} = 0,$$

where $I = \frac{dq}{dt}$ is the current and q is the charge on the capacitor.
  (a) Write a differential equation for the charge q including a Jump function $H(t-T)$ to close the switch at time $t = T$.
  (b) Solve the equation for $V = 10V$, $R = 10K\Omega$, and $C = 100\mu F$ with the switch closing at $T = 1s$ given that $q(0) = 0C$. What is $q(\infty)$? Plot the solution from $t = 0$ to $t = 10$ including a line for $q(\infty)$.
  (c) Repeat the solution for closing at $T = 1s$ and opening at $T = 2s$. Plot the solution from $t = 0$ to $t = 10$ including a line for $q(\infty)$.