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Physics 35100 Mechanics
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Problem Set 6

Question 1. *General Two-Body Central-Force Problem* The general two-body central force Lagrangian for particle 1, mass, m_1, and position $\vec{r}_1$ and particle 2, mass, m_2, and position $\vec{r}_2$ is

$$L = \frac{1}{2}m_1\dot{\vec{r}}_1^2 + \frac{1}{2}m_2\dot{\vec{r}}_2^2 - V(\vec{r}_2 - \vec{r}_1),$$

where $V(\vec{r}_2 - \vec{r}_1)$ is a general central force potential.

(1) Show that the Lagrangian can be separated into a center of mass $\vec{R}$ part and a particle separation part $\vec{r}$:

$$\begin{aligned} L &= L_{CM} + L_s \\ &= \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2 - V(\vec{r}), \end{aligned}$$

where $M\vec{R} = m_1\vec{r}_1 + m_2\vec{r}_2$, $M = m_1 + m_2$, $\mu = m_1m_2/M$ and $\vec{r} = \vec{r}_2 - \vec{r}_1$.

(a) First show that

$$\vec{r}_1 = \vec{R} - \frac{m_2}{M}\vec{r} \text{ and } \vec{r}_2 = \vec{R} + \frac{m_1}{M}\vec{r}.$$

(b) Then plug $\vec{r}_1$ and $\vec{r}_2$ in to L.

(2) Show that the equation of motion for $\vec{R}$ and $\vec{r}$ are

$$\begin{aligned} M\ddot{\vec{R}} &= 0 \\ \mu\ddot{\vec{R}} &= -\vec{\nabla}V(\vec{r}), \end{aligned}$$

where

$$\vec{\nabla}V(\vec{r}) = \frac{\partial V}{\partial \vec{r}} = \frac{\partial V(x,y,z)}{\partial x}\hat{x} + \frac{\partial V(x,y,z)}{\partial y}\hat{y} + \frac{\partial V(x,y,z)}{\partial z}\hat{z}$$

and $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

Question 2. *Hooke's Law in Space* Two masses m_1 and m_2 at positions $\vec{r}_1$ and $\vec{r}_2$ interact via Hooke's law so that their potential $V = 1/2K\vec{r}^2$, where $\vec{r} = \vec{r}_2 - \vec{r}_1$.

(1) Use the general central force equation $\mu\ddot{\vec{r}} = -\vec{\nabla}V$ to find the equations of motion for $\vec{r}$.

(2) Show that the general solution is:

$$\vec{r}(t) = \vec{X}_0\cos(\omega t) + \frac{\vec{V}_0}{\omega}\sin(\omega t).$$

Find the value of ω in terms of other parameters in the problem and explain the meaning of the constant vectors $\vec{X}_0$ and $\vec{V}_0$.

(3) Show that in general the solution $\vec{r}(t)$ lies in a plane for all times t. (Hint: if $\vec{r}(t)$ is always in a plane then there will be a vector $\vec{n}$ which is perpendicular to that plane such that $\vec{r}(t)\cdot\vec{n} = 0$ for all t. Further both $\vec{X}_0$ and $\vec{V}_0$ define the plane that $\vec{r}$ is in.)

(4) Since the solution is in a plane we can always choose that plane to be the x-y plane so that $\vec{r} = x\hat{x} + y\hat{y} + 0\hat{z}$. In this plane, show that the solution orbit is of this form:

$$\vec{r}^T \mathsf{A} \vec{r} = C$$

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = ax^2 + 2bxy + cy^2 = C$$

This is a general quartic equation. Find A and C and show that $\det \mathsf{A} > 0$ and $a > 0$ and $c > 0$ so that the orbit is an ellipse. Hints:

(a) Solve for $\sin(\omega_0 t)$ and $\cos(\omega_0 t)$ and use the fact that $\sin^2 + \cos^2 = 1$.

(b) Notice that:

$$(\omega_0 \vec{X}_0 \times \vec{r})^2 = [\overbrace{(\vec{X}_0 \times \vec{X}_0)}^{0}\omega_0 \cos(\omega_0 t) + (\vec{X}_0 \times \vec{V}_0)\sin(\omega_0 t)]^2$$

$$\omega_0^2 (x_0 y - y_0 x)^2 = (\vec{X}_0 \times \vec{V}_0)^2 \sin^2(\omega_0 t) = (x_0 v_0 - y_0 u_0)^2 \sin^2(\omega_0 t)$$

where $\vec{X}_0 = x_0\hat{x} + y_0\hat{y}$ and $\vec{V}_0 = u_0\hat{x} + v_0\hat{y}$

(c) There is a similar equation for $\cos^2(\omega_0 t)$.

Question 3. *Satellite Orbit* A satellites obits the earth. At the point furthest from the earth it is a distance of 300 km *above* the surface of the earth and has speed of 7.69 km/s.

(1) Find the eccentricity ϵ of the orbit. Use $R_E = 6400$ km for the radius of the earth. (note: $GM_E/R_E^2 = g$, where G is the gravitational constant, M_E is the mass of the earth, and $g = 9.8$ m/s^2 is the value of gravity at the earth's surface.)

(2) Find the minimum distant above the earth and the speed at that point.

Question 4. *Connected Masses* 5 masses, which are connected by ridid mass-less rods to form a rigid structure are at the following positions:

Label	Mass	Position
1	2m	$(0, 0, 0)$
2	2m	$(2l, 0, 0)$
3	2m	$(0, 2l, 0)$
4	2m	$(0, 0, 2l)$
5	m	$(2l, 2l, 2l)$

(1) Without the connecting rods how many degrees of freedom are in the system of 5 masses in 3-dimensions?

(2) How many rod would be needed to connect every possible pair of particles together? (i.e., 1 to 2, 1 to 3, 1 to 4, 1 to 5, 2 to 3, etc.)

(3) If all pairs of the mass are connected by rigid rods, how many degrees of freedom would be left?

(4) What is the minimum number of rods needed to turn the 5 masses into a rigid structure? Give an example connection set. (i.e., 1-2, 2-5, 5-3, etc.) (note: a rod can connect any two particles together so that the distance between them is fixed. For example a rod between particle 1 and 2 would be $2l$ long.)

(5) Find the total mass and the center of mass. How many degrees of freedom does the center of mass represent?

(6) Find the moment of inertia matrix for rotation about the origin. How many degrees of freedom does the moment of inertia matrix represent?

(7) Find the principle axes and the principle moments. Show that the angular momentum vector and the rotational velocity vector are in the same direction for a rotational velocity vector pointing in the direction of one of the principle axes.

(8) Find angular momentum vector for a rotation velocity of $\vec{\omega} = (3\omega_0, 2\omega_0, 4\omega_0)$. Are the angular momentum vector and the rotation velocity vector pointing in the same direction?

(9) Find kinetic energy for a rotation velocity of $\vec{\omega} = (3\omega_0, 2\omega_0, 4\omega_0)$.

Question 5. *Moment of Inertia* Find the moment of inertia matrix of an $a \times a \times b$ solid cuboid of mass M about the origin. The 4 corners of the $a \times a$ face are in the $z = 0$ xy-plane at the points $(0, a/2, 0)$, $(0, -a/2, 0)$, $(a, a/2, 0)$, $(a, -a/2, 0)$. The other four corners are a the points $(0, a/2, b)$, $(0, -a/2, b)$, $(a, a/2, b)$, $(a, -a/2, b)$.