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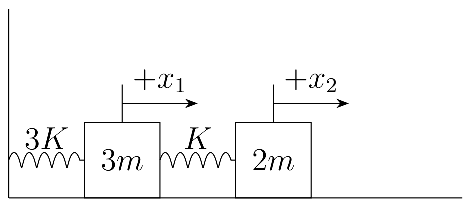
Physics 35100 Mechanics
March 27, 2023

Problem Set 6

From *Classical Mechanics*, R. Douglas Gregory:

Chapter 5: 5.1, 5.16, 5.18

Question 1. *Double Oscillator* Two blocks of mass $m_1 = 3m$ and $m_2 = 2m$ are on a friction-less table. m_1 is connected to a linear spring with spring constant $K_1 = 3K$ and m_2 is connected to a second linear spring with spring constant $K_2 = K$. Both masses are confined to move in one dimension. The masses are initially at rest at their equilibrium positions. x_1 and x_2 measure their displacement from the equilibrium positions.



(1) What is the Kinetic Energy of the system in terms of x_1 and x_2.
(2) What is the Potential Energy of the system in terms of x_1 and x_2.
(3) Show that the potential energy of the system is $2K$ when $x_1 = 1$ and $x_2 = 0$. In this configuration, describe the state of the $3K$ and K springs as stretched, compressed, or at equilibrium.
(4) Find the equations of motion for the system.
(5) Express the equations of motion for the system as a matrix equation of the form $\mathsf{M}\ddot{\mathsf{X}} = -\mathsf{K}\mathsf{X}$, where M and K are 2×2 matrices and X is a 2×1 matrix.
(6) Show that $\mathsf{X}(t) = \mathsf{A}\cos(\omega t + \phi)$, where A is a 2×1 matrix, is a solution to the equation $\mathsf{M}\ddot{\mathsf{X}} = -\mathsf{K}\mathsf{X}$. What are the conditions on A, ω, and ϕ so that $\mathsf{X}(t) = \mathsf{A}\cos(\omega t + \phi)$ is a solution?
(7) Find the values of ω which satisfy $\det(\mathsf{K} - \omega^2\mathsf{M}) = 0$ or $\det(\mathsf{M}^{-1}\mathsf{K} - \omega^2\mathbf{I}) = 0$ for the K and M found above, and $\mathbf{I}$ is the identity matrix.
(8) For each value of ω find an A which satisfies $(\mathsf{K} - \omega^2\mathsf{M})\mathsf{A} = 0$, or $\mathsf{K}\mathsf{A} = \omega^2\mathsf{M}\mathsf{A}$, or $\mathsf{M}^{-1}\mathsf{K}\mathsf{A} = \omega^2\mathsf{A}$.
(9) Using the A's and ω's from above:
  (a) Find the general solution for $\mathsf{X}(t)$.
  (b) Show that in matrix form it can be expressed as:

$$\mathsf{X}(t) = \begin{bmatrix} -2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} C_1 \cos\left(\sqrt{\frac{3}{2}}\omega_0 t + \phi_1\right) \\ C_2 \cos\left(\sqrt{\frac{1}{3}}\omega_0 t + \phi_2\right) \end{bmatrix}$$

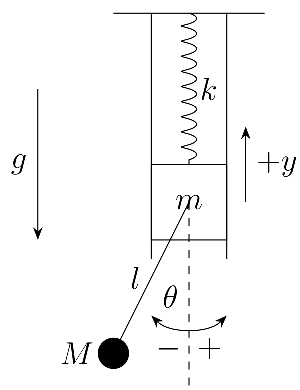
  where C_1 and C_2 are constants.
  (c) Show that it is a solution to the equations of motion.
(10) Show that the change of variables:

$$\mathsf{Y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & 3 \end{bmatrix}^{-1} \mathsf{X}(t)$$

decouples the solution so that $y_1(t)$ and $y_2(t)$ oscillate independently, each with their own frequency and phase. Describe the motion when $C_1 = 1$ and $C_2 = 0$ and when $C_1 = 0$ and $C_2 = 1$.

(11) Find C_1 and C_2 for the initial condition of $\mathsf{X} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\dot{\mathsf{X}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Explain in words what this initial condition represents.

Question 2. *Bouncy pendulum* A simple pendulum of mass M and length l is hanging from a block with mass m that can oscillate at the end of a spring with spring constant k. The block is confined to move only in the vertical direction by two frictionless rails. y measures the vertical displacement of the block from equilibrium, and θ measures the angle of the pendulum from equilibrium.



(1) Find the Lagrangian $L(y, \theta, \dot{y}, \dot{\theta})$ under the assumption that the angle θ is small, so that $\sin\theta \simeq \theta$ and $\cos\theta \simeq 1 - \theta^2/2$. and only retain quadratic terms in $y, \theta, \dot{y}, \dot{\theta}$. (e.g., $y\theta$ is relevant, but $y\theta\dot{\theta}$ is too small. and can be ignored.)

(2) Find the equations of motion. Find the normal modes and corresponding frequencies for the case $m = M = l = 1$, $g = 2$ and $k = 3$, and describe the normal modes.

Useful Equations

(1) Lagrangian in Cartesian coordinate $[\vec{x}_1 ... \vec{x}_N]$:

$$L = T - V = \frac{1}{2} \sum_{n=1}^{N} m_n \dot{\vec{x}}_n^2 - V(\vec{x}_1 ... \vec{x}_N)$$

(2) Generalized coordinates: $\vec{x}_n = \vec{X}_n(q_1 ... q_K)$. K may be less than N if there are constraints.

$$L(q_k, \dot{q}_k) = \frac{1}{2} \sum_{n=1}^{N} m_n \dot{\vec{X}}_n(q_1 ... q_K)^2 - V(\vec{X}_1 ... \vec{X}_N),$$

where

$$\dot{\vec{X}}_n(q_1 ... q_K)^2 = \left[\frac{d\vec{X}_n(q_1 ... q_K)}{dt} \right]^2 = \left[\sum_{k=1}^{K} \frac{d\vec{X}_n(q_1 ... q_K)}{dq_k} \frac{dq_k}{dt} \right]^2 = \left[\sum_{k=1}^{K} \frac{d\vec{X}_n(q_1 ... q_K)}{dq_k} \dot{q}_k \right]^2.$$

(3) Generalized momentum:

$$p_k = \frac{\partial L(q_k, \dot{q}_k)}{\partial \dot{q}_k}.$$

(4) Euler-Lagrange equation of motion:

$$\dot{p}_k = \frac{\partial L(q_k, \dot{q}_k)}{\partial q_k}.$$

(5) General form of 2D coupled harmonic oscillators:

$$\begin{bmatrix} M_1 & M_2 \\ M_2 & M_3 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ q_2 \end{bmatrix} = - \begin{bmatrix} K_1 & K_2 \\ K_2 & K_3 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix},$$

$$\mathsf{M}\dot{\mathsf{Q}} = -\mathsf{K}\mathsf{Q},$$

with general solution:

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \cos(\omega t + \phi),$$

$$\mathsf{Q}(t) = \mathsf{A} \cos(\omega t + \phi).$$

Plugging the solution into the equation motion gives this constraint on ω and A:

$$(\mathsf{K} - \omega^2 \mathsf{M})\mathsf{A} = 0,$$

Eigenvalues ω_k^2 solve this equations:

$$\det(\mathsf{K} - \omega_k^2 \mathsf{M}) = 0,$$

Eigenvectors A_k solve this equations:

$$(\mathsf{K} - \omega_k^2 \mathsf{M})\mathsf{A} = 0,$$

The full solution is:

$$\mathsf{Q}(t) = \sum_k C_k \mathsf{A}_k \cos(\omega_k t + \phi_k),$$

where C_k and ϕ_k are determined by the initial conditions.