One has the relationship[21]

$$\mathbf{j}'=-Ne\mathbf{v}'=c\,\mathrm{curl}\mathbf{M}'+(\partial\mathbf{P}'/\partial t), \quad \text{(B12)}$$

where $\mathbf{P}'$ and $\mathbf{M}'$ represent the electric and magnetic polarization. Since the plasma is nonmagnetic, one has $M'=0$. Using the relationship $\mathbf{P}'=\mathbf{P}\exp i(\mathbf{k}\cdot\mathbf{r}-\omega t)$, one can express the equality (B12) as follows:

$$\mathbf{v}=(i\omega/Ne)\mathbf{P}. \quad \text{(B13)}$$

[21] See, for instance, W. K. H. Panofsky and Melba Phillips, *Classical Electricity and Magnetism* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1955).

Combining (B11) and (B13), one obtains

$$(\omega^2/Ne)\mathbf{P}-[K_BT(\mathbf{k}\cdot\mathbf{P})\mathbf{k}/Nem]+(e/m)\mathbf{E}=0. \quad \text{(B14)}$$

Using the relationship (74), Eq. (B14) can be expressed as

$$(\omega^2/Ne)\mathbf{P}-[s^2(\mathbf{k}\cdot\mathbf{P})\mathbf{k}/3Ne]+(e/m)\mathbf{E}=0, \quad \text{(B15)}$$

or

$$\omega^2\mathbf{P}-(s^2/3)(\mathbf{k}\cdot\mathbf{P})\mathbf{k}+(\omega_1^2/4\pi)\mathbf{E}=0, \quad \text{(B16)}$$

where $\omega_1^2=4\pi Ne^2/m$.

Equation (B16) leads directly to the relationships (79), (83), and (84) given in the text.

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Phase Transition in Elastic Disks*

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The study of a two-dimensional system consisting of 870 hard-disk particles in the phase-transition region has shown that the isotherm has a van der Waals-like loop. The density change across the transition is about 4% and the corresponding entropy change is small.

A STUDY has been made of a two-dimensional system consisting of 870 hard-disk particles. Simultaneous motions of the particles have been calculated by means of an electronic computer as described previously.[1] The disks were again placed in a periodically repeated rectangular array. The computer program has been improved such that about 200 000 collisions per hour can be calculated by the LARC computer regardless of the number of particles in the system. This speed made it possible to follow large systems for several million collisions.

It became necessary to study larger systems in the phase transition region when for smaller ones in three dimensions, it did not seem to be possible for the two phases to exist together in equilibrium.[2,3] Even in the largest three-dimensional system investigated with the improved program (500 hard spheres), the particles were either all in the fluid phase or all in the crystalline phase. The system would typically remain in one phase for many collisions. The occasional shift from one phase to the other would be accompanied by a change of pressure. The equation of state was represented by two disconnected branches overlapping in the density range of the transition, since with the limited number of phase interchanges it was not possible to average the two branches.

Two-dimensional systems were then studied, since the number of particles required to form clusters of particles of one phase of any given diameter is less than in three dimensions. Thus, an 870 hard-disk system is effectively much larger than a 500 hard-sphere system. First, however, it was necessary to establish that small two-dimensional systems behave analogously to the three-dimensional systems. This is illustrated in Fig. 1 by the two disconnected branches drawn lightly through the triangular points for a 72-particle system. In that figure, the reduced pressure pA_0/NkT is plotted against the reduced area A/A_0, where A_0 is the area of the system at close packing. In the region of A/A_0 from 1.33 to 1.35 the system fluctuated infrequently between a high-pressure fluid branch and a low-pressure crystalline branch, while at A/A_0 of 1.31 and higher densities the solid phase was always stable.

For the larger 870-particle system, however, the two phases exist side by side. One piece of evidence for this coexistence is the cathode-ray tube pictures described earlier (see Fig. 2).[1] The trajectories of the particles plotted on the oscilloscope show regions where the particles are localized (crystallites) in between regions of mobile particles (fluid). Further evidence is the characteristically large pressure fluctuations in the phase transition region where two states can exist with almost equal probability. The extent of the fluctuations in a typical run of about 10 million collisions is obtained

* This work was performed under the auspices of the U. S. Atomic Energy Commission.

[1] B. J. Alder and T. E. Wainwright, J. Chem. Phys. **31**, 459 (1959).

[2] B. J. Alder and T. E. Wainwright, J. Chem. Phys. **33**, 1439 (1960).

[3] W. W. Wood, R. R. Parker, and J. P. Jacobson, Suppl. Nuovo cimento **9**, 133 (1958).

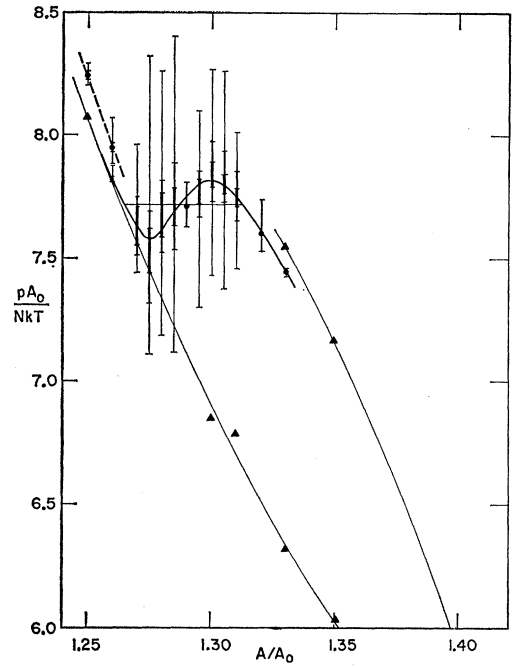

FIG. 1. The equation of state of hard disks in the phase transition region.

by breaking the run into intervals of 50 thousand collisions (about 100 collisions for each particle) and determining the average pressure in each interval. The light vertical lines at various densities (Fig. 1) extend from the maximum to the minimum pressure found among these intervals. As can be seen, the fluctuations are much larger in the phase transition region than in the pure solid ($A/A_0<1.26$) and pure fluid ($A/A_0>1.33$) phases and they are also larger in the middle of the phase transition region than near the ends. The medium vertical lines in Fig. 1 indicate the middle range of the fluctuations, that is, $\frac{1}{4}$ of the intervals show pressures above the top of the vertical line and $\frac{1}{4}$ show pressures below the bottom of the line. For the shorter runs at A/A_0 of 1.32 and 1.29 only this medium vertical line could be drawn.

The heavy vertical lines in Fig. 1 indicate the estimated accuracy of the average pressure determination at each density. These estimates are made by comparing several runs of 10 million collisions each with various starting conditions. The comparison was typically within 1% at a few selected densities. The fact that the pressures calculated with varying starting conditions agree rather closely is the only indication that phase space has been adequately sampled. Four different starting conditions have been used: (1) all the particles are located in lattice positions with only one particle in motion; (2) all particles are in lattice positions and in motion with randomly selected velocities; (3) the starting configuration for a run at one density is taken from an instantaneous configuration at a lower density, having been effectively changed by increasing the diameter of the particles; and (4) the same as (3) except a higher density configuration is used as an initial low-density one.

The smooth curve drawn through the heavy vertical sections in Fig. 1 clearly shows a van der Waals loop-like behavior for the equilibrium state of a finite system. To confirm this it was found that decreasing the density from A/A_0 of 1.29 to 1.30 by procedure (4) above, increased the average pressure. Similarly, increasing the density from A/A_0 of 1.285 to 1.280 and subsequently to 1.275 by procedure (3) decreased the average pressure in both cases, although, immediately after increasing the density, the pressure at 1.275 was higher for some millions of collisions. This shows that the density must be increased very slowly near the solid region or otherwise the particles will be locked into a disordered configuration. Thus, on increasing the density by procedure (3) from A/A_0 of 1.275 to 1.26 the crystalline region of phase space (lower value) became disconnected from the disordered region (upper value) as seen in Fig. 1, since the extremes of the pressure fluctuations no longer overlap. A further increase in density of the disordered or glass-like configuration at A/A_0 of 1.26 to 1.25 defines the dashed line in Fig. 1.

It has been shown that in an infinite system the isotherms for this system would always have to be of negative or at most zero slope.[4] Thus, the loop could not exist. The existence of a loop for finite systems probably derives from the fact that the constraint of constant density is imposed over a region occupied by however many particles are dealt with. This constraint restricts

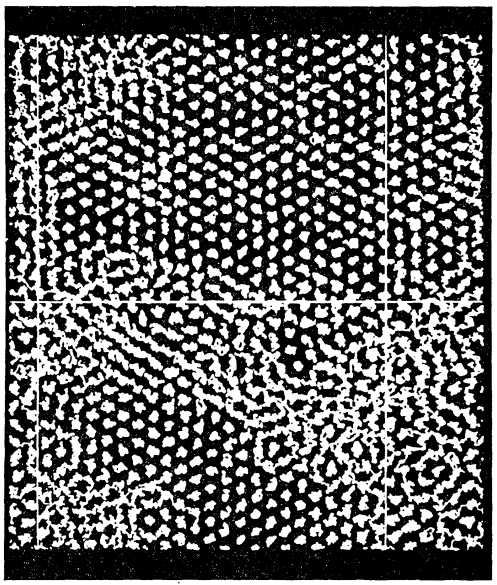

FIG. 2. The traces of the centers of particles in the phase-transition region showing fluid and crystalline regions. The horizontal and vertical lines represent an arbitrary grid.

[4] L. van Hove, Physica **15**, 951 (1949).

the configurations which can be reached in a finite system, that is, for example, a fluid in equilibrium with crystallites of average size greater than the number of particles dealt with is impossible to achieve. This constraint for the small systems previously investigated resulted in stabilization of the predominant phase. Thus, the system was either all solid and when a rare fluctuation disordered enough of the system, it became completely fluid. For the 870-particle system the constraint again stabilizes the more abundant phase causing the pressure to be high on the fluid side and low on the crystal side. It thus seems that the phase separation which might occur in infinite systems is not complete in finite systems, since a sizable portion of the system lies in the fluid-crystal boundary region and this region is of intermediate density and evidently takes on more of the character of the predominant phase.

The horizontal line in Fig. 1 drawn at $pA_0/NkT=7.72$ and extending from A/A_0 of 1.266 to 1.312 corresponds to the usual "equal area" rule. If the phase transition for an infinite system is of first order at the pressure indicated by this straight line, then the resulting entropy change across the transition $\Delta S/Nk$ is $p\Delta A/NkT=0.36$. The change of entropy across the same density interval corresponding to the expansion of the one particle cell as calculated by the free volume theory is 0.30. This indicates that the change of communal entropy (0.06) across the transition is very much smaller than unity. This is hardly in accord with the view[5] that the difference between a dense fluid and a solid is one of the accessibility of the entire space in the fluid and localization of a molecule in a solid.

The complete equation of state and comparisons of it with the predictions of various theories will be the subject of further publications.

We are deeply indebted to Mary Ann Mansigh and Norman Hardy for their invaluable help in programming, and to Dr. Sidney Fernbach of the Livermore Computing Division for his cooperation.

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Thermal Conduction in Liquid Helium II. I. Temperature Dependence

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The thermal conductivity of liquid helium contained in a cylindrical stainless-steel capillary 0.080 cm in diam by 5.16 cm long has been studied between 0.9°K and the λ point. The relation between temperature gradient and heat current density W for heat currents greater than the critical heat current density W_c is best expressed in the form $\text{grad}T=DW^n$, where D is a temperature-dependent constant and n varies from about 3.0 at low temperatures to about 3.5 above 1.7°K. Below W_c the temperature gradient is much smaller, and is determined entirely by the viscosity of the normal component. W_c was measured over the entire temperature range by a combination of two methods, which are in complete agreement in the region of overlap. The results suggest that, below about 1.7°K, W_c is the result of some sort of normal turbulence describable by a Reynolds number involving the normal fluid velocity but the *total* density. At higher temperatures such an explanation is no longer adequate, and some other type of critical velocity must be invoked.

I. INTRODUCTION

LIQUID helium II has an unusually high thermal conductivity, which, in terms of the two-fluid model, can be explained by an internal convection of the normal and superfluid components. According to this model, normal fluid flows away from the source of heat with a velocity $\mathbf{v}_n$ which is related to the heat current density $\mathbf{W}$ by the equation

$$\mathbf{v}_n=\mathbf{W}/\rho ST, \tag{1}$$

where ρ is the density, S the entropy per unit mass, and T the temperature. The superfluid flows in the opposite direction with a velocity $\mathbf{v}_s$ which is determined by the additional condition that there be no net mass flow,

$$\rho_n\mathbf{v}_n+\rho_s\mathbf{v}_s=0, \tag{2}$$

where ρ_n and ρ_s are, respectively, the normal and superfluid densities.

As long as $\mathbf{W}$ is sufficiently small, the only dissipative mechanism present is the viscosity of the normal component η_n. For a cylindrical channel of radius r, the relation between temperature gradient and heat current density is then given by [1,2]

$$\mathbf{W}=-(r^2\rho^2S^2T/8\eta_n)\,\text{grad}T. \tag{3}$$

* Operated with support from the U. S. Army, Navy, and Air Force.

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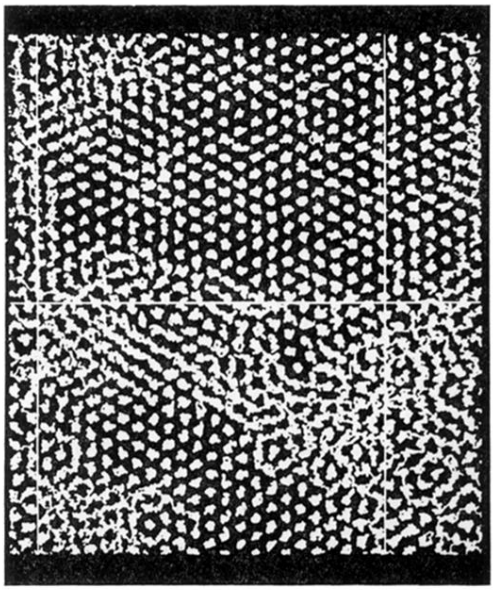

FIG. 2. The traces of the centers of particles in the phase-transition region showing fluid and crystalline regions. The horizontal and vertical lines represent an arbitrary grid.