

Topic E

Designing motion rather than predicting it

Question and starting model

What torque is needed to produce a prescribed pendulum motion, and what makes one motion preferable to another? This is an inverse-dynamics problem: the trajectory is chosen and the actuator torque is inferred.

Use a point mass m on a massless rigid rod of length ℓ , in uniform gravity g . The pivot is fixed and friction is absent. Measure θ from the downward vertical. A controllable pivot torque $\tau(t)$ acts in the direction of increasing θ . The ideal actuator has no efficiency or saturation model unless one is explicitly added.

Move from $\theta_0 = 0$ to $\theta_1 = \pi/2$ in a duration T_f , with zero initial and final angular velocities. Derive the reduced equation of motion and use it to recover the torque for a prescribed smooth trajectory. Use $I = m\ell^2$ for the moment of inertia and keep L for the Lagrangian.

A bounded family you can begin with

Set $s = t/T_f$, where $0 \leq s \leq 1$, and consider

$$\theta(t) = \theta_0 + (\theta_1 - \theta_0)f_a(s),$$

$$f_a(s) = 10s^3 - 15s^4 + 6s^5 + as^3(1-s)^3.$$

Check the position, velocity, and acceleration endpoint conditions rather than assume them. Start with $a = 0$, then compare a small set such as $a = -6, 0, 6$. Check whether the motion is monotonic for your selected shapes. The zero endpoint accelerations do not imply zero holding torque at an endpoint in gravity.

What you need to start

Generalized coordinates, virtual work/generalized torque, the pendulum equation, work–energy, and basic numerical integration or plotting. Neither feedback-control theory nor optimization software is required.

First-week task

Verify the endpoint conditions for $a = 0$, derive its torque history, and give one check of the result. State a candidate comparison measure. This first calculation should make the proposal concrete before studying a family of trajectories.

Bounded investigation

Compare the selected shapes at the *same* endpoints and duration. Choose and justify one primary performance measure, for example

$$\max_{0 \leq t \leq T_f} |\tau(t)|, \quad W_+ = \int_0^{T_f} \max(\tau \dot{\theta}, 0) dt, \quad \text{or} \quad \int_0^{T_f} \tau(t)^2 dt.$$

These are different quantities with different meanings and units. The last is an effort measure, not automatically energy consumption. If using W_+ , explain what the idealized actuator is assumed to do during negative power.

Then examine how the same comparison changes at a few durations. A convenient scale is $t_g = \sqrt{\ell/g}$, with T_f/t_g between approximately 1 and 3. Nondimensional values $m = \ell = g = 1$ are acceptable when the physical scaling is stated. A small set of shapes and durations is enough; global optimal control is not the task.

The work question you must address

Derive the signed actuator work from the mechanical energy balance. Determine whether fixed initial and final states leave signed net work available as a meaningful discriminator among trajectories. Explain why lower peak torque, lower positive work, and lower signed net work are not interchangeable claims.

State whether your comparison includes a final holding interval. The default investigation ends at T_f ; do not silently add different holding costs for different motions.

Validation

Substitute the inferred torque into the forward equation and check that it recovers the intended motion, analytically or numerically. Independently check work–energy. A numerical implementation should demonstrate refinement of the integral or peak estimate at the accuracy claimed, not merely plot a smooth torque curve.

Explain how gravity and inertia compete as duration changes. Define the actuator assumptions before interpreting any measure as physically useful.

Stopping rule

A complete project has one mechanical model, a bounded family of prescribed trajectories, a justified comparison measure, a small duration study, and both a forward-equation and energy check. It concludes what is preferable *under the stated measure*, not universally.

Optional extension, only after completion

Add one torque limit, damping term, or alternative endpoint condition. Feedback, impacts, full robotics, and actuator-efficiency modeling are not required.

Starting references. Generalized-coordinate and work–energy course notes; the resources sheet includes an optional standard dynamics reference. Required trajectory definitions are supplied here.

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