

Topic D

Beyond the Kepler ellipse

Question and starting model

How does a specified deformation of a central potential change orbital closure and the angle between successive periapses? Separate a physical change in the orbit from numerical precession.

Begin with the exact reduced two-body problem of mass $\mu > 0$, radial coordinate $r > 0$, and planar angle θ . Use

$$L = \frac{\mu}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - V(r), \quad V(r) = -\frac{k}{r} + \frac{\beta}{r^2}, \quad k > 0.$$

L denotes the Lagrangian; use $J = \mu r^2 \dot{\theta} > 0$ for the conserved scalar angular momentum. The radial energy and reconstruction relation are

$$\frac{\mu}{2}\dot{r}^2 + V_{\text{eff}}(r) = E, \quad V_{\text{eff}}(r) = V(r) + \frac{J^2}{2\mu r^2}, \quad \dot{\theta} = \frac{J}{\mu r^2}.$$

This is a classical model potential. It is not, by itself, a relativistic model of planetary precession.

What you need to start

Two-body reduction, angular momentum, effective potentials, the Kepler reference orbit, and basic ODE integration or numerical quadrature. No canonical perturbation theory is needed. The topic supplies the reduced starting equations so that a later lecture is not a barrier to beginning.

Required foundation

Reproduce a noncircular bound orbit for $\beta = 0$. Define *periapsis* as a local minimum of $r(t)$ and choose a positive orientation for θ . Decide whether the comparison holds E, J fixed or holds physical initial data fixed. Explain the distinction; changing β at the same initial position and velocity generally changes E .

Derive conditions ensuring that the chosen orbit remains bound and avoids the origin for the parameters studied. The total V_{eff} , not the centrifugal term alone, controls the small- r barrier. Exclude circular cases from a measurement that requires distinct periapses.

First-week task

Produce one checked Kepler reference orbit or radial quadrature, and define an observable for the angular separation of successive periapses. Bring a preliminary result and a clear choice of comparison data to the proposal.

Bounded investigation

Use $u(\theta) = 1/r(\theta)$ to obtain an orbital equation for the specified potential. Determine the angular interval $\Delta\theta$ between successive periapses and define signed precession per radial period as $\Delta\theta - 2\pi$.

Choose a small set of β values, including zero, for which your specified noncircular bound motion exists. Compare the analytical prediction with numerical integration of the original time-domain equations or with an independent radial quadrature. State the units or nondimensionalization.

Measure angle continuously rather than reducing every value modulo 2π . Decide how periapses are located and estimate the sensitivity to timestep, solver tolerance, or quadrature accuracy. Do not integrate only the orbital equation you just solved and call that an independent physical check.

A feasible parameter convention

One convenient option is $\mu = k = r_0 = 1$, $\dot{r}_0 = 0$, and initial tangential speed $v_0 = 0.8$. You may explore, for example, $\beta = -0.08, 0, 0.08$, after checking that each selected orbit satisfies your conditions. These are dimensionless benchmark choices, not compulsory data.

For Cartesian integration, derive the force from the stated potential. For radial quadrature, account for turning-point endpoint behavior instead of integrating a singular expression naively. Either independent route is sufficient; doing both is an optional extension.

Validation and interpretation

Recover the Kepler limit, check energy and angular momentum where appropriate, and demonstrate that the measured angular shift is insensitive to numerical refinement at the accuracy claimed. Explain its sign and dependence on your dimensionless deformation parameter.

Distinguish closure after one radial period from closure after several periods. A special parameter value may produce a closed non-Kepler orbit; do not infer a universal nonclosure statement from a few pictures.

Stopping rule

A complete project has one fixed potential family, a bound-orbit domain, an analytical precession prediction, and a controlled independent comparison over a few parameters. It explains the physical and numerical distinctions. General central-potential classification, three-body motion, or general relativity is not required.

Optional extension, only after completion

Study one limiting approach to a circular orbit or compare fixed initial data with fixed E, J . More potential families are unnecessary for a complete investigation.

Starting references. Central-force course notes and Homework 5's Kepler baseline. The resources sheet gives optional reading and numerical documentation.

[Topic chooser](#) [Common assignment](#) [Resources](#)