

Topic C

From mechanisms to soft modes

Question and starting model

How does changing a small framework turn free internal motion into a restoring mode? Separate the effects of connectivity, reference geometry, spring stiffness, and supports rather than treating every zero eigenvalue as the same phenomenon.

Take four planar nodes at

$$\mathbf{r}_1 = (0, 0), \quad \mathbf{r}_2 = (a, 0), \quad \mathbf{r}_3 = (a, a), \quad \mathbf{r}_4 = (0, a), \quad a > 0.$$

Connect the four sides, with unstretched springs of stiffness $\kappa > 0$. Compare this unbraced square with a square containing the additional diagonal 1–3. Work first in the small-displacement approximation with no prestress. Stack the displacements as

$$\mathbf{u} = (u_{1x}, u_{1y}, u_{2x}, u_{2y}, u_{3x}, u_{3y}, u_{4x}, u_{4y})^T.$$

For an edge from i to j , define

$$\hat{\mathbf{n}}_{ij} = \frac{\mathbf{r}_j - \mathbf{r}_i}{|\mathbf{r}_j - \mathbf{r}_i|}, \quad e_{ij} = \hat{\mathbf{n}}_{ij} \cdot (\mathbf{u}_j - \mathbf{u}_i).$$

Assemble these rows into $\mathbf{e} = A\mathbf{u}$. With positive spring stiffnesses in the diagonal matrix C ,

$$V = \frac{1}{2} \mathbf{e}^T C \mathbf{e}, \quad K = A^T C A, \quad \mathbf{F}_{\text{int}} = -K\mathbf{u}.$$

Here A includes reference geometry; it is not just a scalar graph-incidence matrix. K is a stiffness matrix, not the Lagrangian.

What you need to start

The course's compatibility, constitutive response, and force balance, nullspaces and eigenvalues, and small-displacement linearization. MATLAB or NumPy may be used for the matrix calculations. No general rigidity theorem or topology course is required.

Required foundation and first-week task

Construct A explicitly for the unbraced and braced square. Identify the two translations and infinitesimal rotation in the free framework. Separate these rigid motions from any internal zero mode.

For the proposal, bring the unbraced matrix, at least one directly checked row, and a candidate null vector with a physical interpretation. You need not finish the entire eigenvalue study before proposing the question.

Bounded investigation: choose one branch

First compare the free unbraced and braced baselines. Then choose **one** investigation:

1. **A brace becoming weak.** Set the diagonal stiffness to $\alpha\kappa$. Fix node 1 in both directions and constrain the vertical displacement of node 2. Eliminate the constrained coordinates before forming the reduced matrices. Study the lowest nonzero stiffness eigenvalue as $\alpha \rightarrow 0^+$, using a prediction from the mechanism direction and a small numerical comparison. Equal nodal masses may convert stiffness eigenvalues to frequencies, but a full general normal-mode theory is unnecessary.
2. **Supports versus internal rigidity.** Compare two explicitly specified support choices on the same free-framework baselines. Predict which rigid motions or internal mechanisms each support removes, then test the predictions. Explain how a support can remove a zero mode without changing the internal connectivity.

Do not study both branches merely to make the report longer. Define a focused question within the selected branch and state what quantities are held fixed.

Validation

For positive C , use the energy identity to check that $\ker(A^T C A) = \ker A$. Verify representative null vectors directly, not only through a numerical routine's small eigenvalues. State the numerical tolerance used to identify zero and inspect its sensitivity.

For one displacement pattern, compare $A\mathbf{u}$ with exact finite edge-length changes at successively smaller amplitudes. An infinitesimal zero mode need not make a straight, finite-amplitude displacement preserve every length exactly. Distinguish first-order compatibility from a claim of an exact finite mechanism.

Stopping rule

A complete project contains the two baseline matrices and nullspace interpretations, one of the two bounded investigations, and an independent check. Explain which conclusion follows from geometry, stiffness, or supports. Larger networks and a general classification of rigidity are not required.

Optional extension, only after completion

Study one state of self-stress, one slightly perturbed geometry, or an exact finite mechanism. Prestressed-framework dynamics and large periodic lattices are beyond the required scope.

Starting references. Lecture 2 and the coupled-system material. The edge-extension definition on the first page is the necessary two-dimensional starting rule. Ask about matrix rank or small-displacement linearization before expanding the model.

[Topic chooser](#) [Common assignment](#) [Resources](#)