

Topic B

Conservation before the continuum limit

Question and starting model

Which conservation laws belong exactly to a discrete mechanical model, and which are only approximately respected by an approximation to a continuous system? Derive a discrete symmetry result, then use a controlled comparison to find out.

Use two equal masses m with planar positions $\mathbf{x}_1, \mathbf{x}_2$ and an isolated central interaction. A convenient nonsingular baseline is

$$V(\mathbf{x}_1, \mathbf{x}_2) = \frac{\kappa}{2} |\mathbf{x}_2 - \mathbf{x}_1|^2, \quad \kappa > 0.$$

This idealized spring has zero rest length. Its usefulness here is the exact rotational and common-translation symmetry, not realism at contact. Let $\mathbf{q} = (\mathbf{x}_1, \mathbf{x}_2)$. For a fixed step h , start from

$$L_d(\mathbf{q}_n, \mathbf{q}_{n+1}) = \sum_{i=1}^2 \frac{m}{2h} |\mathbf{x}_{i,n+1} - \mathbf{x}_{i,n}|^2 - hV(\mathbf{q}_n).$$

The action is the sum of these edge contributions. Fixed endpoints are used to derive the equations of motion. Symmetry variations need not keep endpoints fixed.

What you need to start

Discrete action, localized variation, and the translation/rotation examples of Noether's theorem. Matrix Lie groups and canonical transformations are not prerequisites. A planar cross product here means the signed scalar $\mathbf{a} \wedge \mathbf{b} = a_x b_y - a_y b_x$.

For clarity, partial derivatives of $L_d(\mathbf{a}, \mathbf{b})$ below are column vectors with respect to its first or second argument. Define the incoming discrete momentum at node n by

$$\mathbf{p}_n = \left. \frac{\partial L_d(\mathbf{a}, \mathbf{b})}{\partial \mathbf{b}} \right|_{(\mathbf{a}, \mathbf{b}) = (\mathbf{q}_{n-1}, \mathbf{q}_n)}.$$

Do not replace this quantity silently by a centered finite-difference velocity.

Required foundation and first-week task

Vary one interior node to derive the discrete Euler–Lagrange equation. Differentiate the invariance of L_d under common translation or common rotation. Combine the two identities to obtain the associated conserved charge.

For the proposal, bring the derived update and a checked symmetry calculation for at least one of these transformations. This is already executable mathematical work; computation may be added afterward.

Bounded investigation

Apply the discrete conservation result to the two-particle baseline, identifying total linear momentum and signed planar angular momentum with their correct discrete definitions. Choose **one** comparison:

1. **Change the numerical method, not the physical system.** Compare the action-derived update with ordinary forward Euler,

$$\mathbf{x}_{i,n+1} = \mathbf{x}_{i,n} + h\mathbf{v}_{i,n}, \quad m\mathbf{v}_{i,n+1} = m\mathbf{v}_{i,n} - h \frac{\partial V}{\partial \mathbf{x}_i}(\mathbf{q}_n).$$

Determine separately what happens to total linear and angular momentum. Do not assume that a generic method loses every conservation law.

2. **Change the physical model.** Add one specified anisotropic interaction term or external potential. Identify exactly which symmetry is broken, derive the resulting balance/change equation, and explain which other conservation law survives.

State how the initial discrete data are chosen. For the action-derived map, the incoming-momentum definition provides a transparent node state and avoids ambiguity about backward, forward, and centered velocities.

Validation

Verify the conserved quantity directly in the recurrence or first-order update. The comparison may be fully analytical when the finite-step charge change can be derived explicitly. A computational version should compare the analytical charge-change prediction with trajectories and report the effects of smaller timestep and numerical precision.

Keep physical symmetry breaking, discretization error, and floating-point error distinct. Exact momentum conservation does not automatically imply exact physical-energy conservation for a fixed-step variational method.

Stopping rule

A complete project has one discrete symmetry-to-conservation derivation, the two-particle application, and one controlled comparison with a direct verification. It states what is exactly conserved, what changes, and why. A theorem for general fields or a general theory of discrete energy is not required.

Optional extension, only after completion

Consider an invariant with a nonzero boundary term, or relate the incoming/outgoing momenta to later Hamiltonian concepts. Neither is necessary for the required project.

Starting references. Lectures 4 and 7; optional selected passages from Marsden and West on discrete Euler–Lagrange and discrete Noether structure. The resources sheet distinguishes essential starting material from advanced reading.

[Topic chooser](#) [Common assignment](#) [Resources](#)