

Topic A

Discrete evolution: accuracy versus preserved structure

Question and starting model

What changes when a simple oscillator update is symmetrized? Investigate trajectory accuracy, phase, energy behavior, and reversibility without assuming that one method is better by every measure.

Use the harmonic oscillator $\ddot{x} = -\omega^2 x$, with $m > 0$, $\omega > 0$, and timestep $h > 0$. Work with the numerical state $z_n = (x_n, v_n)^T$. Define drift and kick maps by

$$D_h = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}, \quad T_h = \begin{pmatrix} 1 & 0 \\ -h\omega^2 & 1 \end{pmatrix}.$$

Matrix products act from right to left. Compare

$$Q_h = T_h D_h \quad (\text{move then turn}), \quad S_h = D_{h/2} T_h D_{h/2}.$$

These are numerical approximations to the same oscillator initial-value problem, not the exact finite-time flow. The physical energy for the comparison is

$$E(x, v) = \frac{1}{2}mv^2 + \frac{1}{2}m\omega^2 x^2.$$

What you need to start

Lecture 4's step/turn description, matrix multiplication and eigenvalues, the oscillator solution, and basic MATLAB or Python/NumPy. No Poisson brackets, matrix logarithms, or general symplectic theory are required. In the earlier step convention, $s = hv$; state carefully how a step or velocity is attached when translating between conventions.

Required foundation

Derive the two matrices and their determinants. Define reversibility by $M(-h) = M(h)^{-1}$ and test it. Find the oscillatory/stable parameter range from the eigenvalues, checking marginal endpoints separately. Establish an exact reference solution for the same physical $x(0), \dot{x}(0)$.

Initialize both numerical states with the same physical initial data when measuring initial-value error. A change of coordinates or staggering may be useful, but must not quietly change the initial condition being compared.

First-week task

Implement Q_h and S_h for one timestep size, evolve the same nonzero initial state, and compare with the exact solution over several periods. Confirm that your code implements the displayed matrix order. Bring a checked matrix or one interpretable reference comparison to the proposal.

Bounded investigation

Use one original map and one symmetric map; the pair above is the default. State a focused question about what symmetrization changes. Your comparison must distinguish:

1. **Trajectory and phase.** Measure state error over a fixed physical interval at three or more successively smaller timesteps. Nondimensionalize position and velocity before combining their errors. Determine the eigenvalue phase advance per step and compare the resulting numerical frequency with ω .
2. **Long-time physical energy.** Compare energy error over many periods at a fixed stable timestep, then repeat at a smaller timestep. Explain whether an error is bounded, growing, or oscillatory and whether that statement is analytical or observed over a finite run.

Compare runs over equal physical durations, not merely equal step counts. Avoid measuring error only at a specially chosen time at which cancellation could hide it; a maximum over a fixed interval is useful. You need not tune an optimal method or examine many integrators.

Validation and interpretation

Check the matrices directly against sequential drift/kick updates, verify convergence against the exact solution, and compare a predicted phase advance with measured motion. A bounded-looking energy plot is not proof of an exact invariant. Any claimed invariant $z^T G z$ should satisfy $M^T G M = G$ algebraically.

Explain separately what determinant one, reversibility, phase advance, and physical-energy behavior establish. Do not infer that preserving area means preserving physical energy, or that higher trajectory accuracy guarantees improvement in every phase measure.

Stopping rule

The project is complete with the derived matrices and stability/reversibility tests, a fixed-time convergence study, a phase comparison, and a validated long-time energy comparison, together with a clear conclusion answering your question. One well-chosen pair of methods is sufficient.

Optional extension, only after completion

Choose at most one: derive an exact quadratic invariant; add explicit Euler as a contrast; or interpret the maps using later Hamiltonian splitting ideas. Nonlinear oscillators, higher-order methods, and shadow-Hamiltonian expansions are not required.

Starting references. Lecture 4 and the oscillator material; the resources sheet points to optional geometric-integration reading. Required definitions are supplied here.

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