

Problem Set 6

PHYS V1100: Analytical Dynamics

Fall 2026

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This set closes the central-force unit and develops the course transition from local quadratic dynamics to nonlinear phase space and rigid-body motion. The emphasis is structural: choose representations that expose the mechanics, distinguish local predictions from global behavior, and use independent checks before trusting algebra or computation.

Show enough reasoning that another person can follow your argument. In the small-oscillation problem, keep the distinction between Euclidean and mass-weighted orthogonality explicit. In the nonlinear problem, make the analytical phase-space prediction before computing. In the rigid-body problems, distinguish the reference point matrix, the moving frame, and body- versus laboratory-frame component lists. MATLAB is the primary computational language; Python/NumPy is also acceptable where computation is requested. In the written-plus-oral problem, all students complete the written portion; only the designated presenter prepares the oral extension.

Problem 1

A different central force: the isotropic harmonic potential. After the center-of-mass motion of a two-body system has been separated, let the relative coordinate $\mathbf{r}$ have reduced mass μ and potential

$$V(r) = \frac{1}{2}kr^2, \quad k > 0, \quad r = \|\mathbf{r}\|.$$

The initial data are

$$\mathbf{r}(0) = \mathbf{r}_0, \quad \dot{\mathbf{r}}(0) = \mathbf{v}_0.$$

Do not rederive the center-of-mass transformation.

- (a) Derive the relative equation of motion and show that

$$\mathbf{r}(t) = \mathbf{r}_0 \cos(\omega t) + \frac{\mathbf{v}_0}{\omega} \sin(\omega t), \quad \omega = \sqrt{\frac{k}{\mu}}.$$

Use this solution to explain why the motion is planar and identify the conserved angular-momentum vector $\mathbf{J}$.

- (b) Choose Cartesian axes in the orbital plane and define

$$B = \begin{pmatrix} x_0 & u_0/\omega \\ y_0 & v_0/\omega \end{pmatrix}, \quad \mathbf{s}(t) = \begin{pmatrix} \cos \omega t \\ \sin \omega t \end{pmatrix},$$

where $\mathbf{r}_0 = (x_0, y_0)^T$ and $\mathbf{v}_0 = (u_0, v_0)^T$. Show that $\mathbf{r} = B\mathbf{s}$. If $\det B \neq 0$, eliminate time and show that the orbit is an ellipse centered on the force center.

- (c) Interpret the exceptional case $\det B = 0$. Show that it is precisely the zero-angular-momentum case and describe the orbit.

- (d) Both the attractive inverse-square force and the isotropic harmonic force can produce closed elliptical bound orbits. State one essential geometric difference between the two ellipses, and explain why that difference is physically meaningful.

Problem 2

General small oscillations: mass weighting and normal coordinates. Two blocks move on a horizontal frictionless line. A mass m is attached to a wall by a spring of stiffness $3k$, and a mass $2m/3$ is attached to the first mass by a spring of stiffness k . Let q_1 and q_2 be the signed extensions of the two springs measured from equilibrium. Then the physical displacements of the masses are

$$y_1 = q_1, \quad y_2 = q_1 + q_2.$$

Define $\mathbf{q} = (q_1, q_2)^T$ and $\omega_0^2 = k/m$.

- (a) Starting from the particle kinetic energy and the spring energies, show that

$$L^{(2)} = \frac{1}{2} \dot{\mathbf{q}}^T M \dot{\mathbf{q}} - \frac{1}{2} \mathbf{q}^T K \mathbf{q},$$

with

$$M = \frac{m}{3} \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}, \quad K = k \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}.$$

Explain what this coordinate choice has simplified and what it has made less simple.

- (b) Derive

$$M \ddot{\mathbf{q}} + K \mathbf{q} = 0$$

and seek a normal mode $\mathbf{q} = \mathbf{u} a(t)$ with $\ddot{a} + \omega^2 a = 0$. Solve

$$K \mathbf{u} = \omega^2 M \mathbf{u}$$

to find the two frequencies and one convenient unnormalized mode shape for each.

- (c) Verify that the two mode shapes are not Euclidean-orthogonal but are M -orthogonal:

$$\mathbf{u}_1^T M \mathbf{u}_2 = 0.$$

Find normalization constants so that the normalized vectors $\hat{\mathbf{u}}_\alpha$ satisfy

$$\hat{\mathbf{u}}_\alpha^T M \hat{\mathbf{u}}_\beta = \delta_{\alpha\beta}.$$

- (d) Let

$$U = (\hat{\mathbf{u}}_1 \quad \hat{\mathbf{u}}_2), \quad \mathbf{q} = U \mathbf{a}.$$

Show that

$$U^T M U = I_2, \quad U^T K U = \text{diag}(\omega_1^2, \omega_2^2),$$

and hence write the quadratic Lagrangian in the normal coordinates a_1, a_2 .

- (e) Suppose initially the first mass is at equilibrium, the second is displaced a distance d to the right, and both are released from rest:

$$y_1(0) = 0, \quad y_2(0) = d, \quad \dot{y}_1(0) = \dot{y}_2(0) = 0.$$

Determine $\mathbf{q}(0)$. Use the mass-weighted projection

$$a_\alpha(0) = \hat{\mathbf{u}}_\alpha^T M \mathbf{q}(0)$$

to show that both normal modes are excited. Explain why the subsequent motion is generally not a single-frequency oscillation.

Problem 3

Nonlinear phase portraits: what the quadratic approximation cannot see. Consider first the exact simple pendulum

$$\ddot{\theta} + \omega_0^2 \sin \theta = 0, \quad \omega_0^2 = \frac{g}{\ell}.$$

Its energy is

$$E = \frac{1}{2} m \ell^2 \dot{\theta}^2 + m g \ell (1 - \cos \theta).$$

- (a) Introduce the dimensionless velocity $u = \dot{\theta}/\omega_0$ and dimensionless energy $\varepsilon = E/(mg\ell)$. Show that the phase curves satisfy

$$\boxed{\frac{1}{2} u^2 + 1 - \cos \theta = \varepsilon.}$$

Use this equation to classify the motion for

$$0 < \varepsilon < 2, \quad \varepsilon = 2, \quad \varepsilon > 2.$$

Sketch the corresponding qualitative phase portrait and identify the separatrix.

- (b) Linearize the pendulum equation about the stable equilibrium $\theta = 0$. What local behavior does the linear theory predict correctly? Name at least two global features of the exact phase portrait that the linearized equation cannot reveal.
- (c) Now compare the one-dimensional potentials

$$V_+(x) = \frac{\lambda}{4} x^4, \quad V_-(x) = -\frac{\lambda}{4} x^4, \quad \lambda > 0,$$

with unit mass. Show that both have the same equilibrium and the same quadratic linearization at $x = 0$, yet one equilibrium is stable and the other unstable. Explain what this demonstrates about a zero eigenvalue of the Hessian.

- (d) **Focused computation.** Work in dimensionless time $\tau = \omega_0 t$, so

$$\frac{d^2 \theta}{d\tau^2} + \sin \theta = 0.$$

Numerically integrate the two initial states

$$\text{A: } \theta(0) = 1.5, \quad \frac{d\theta}{d\tau}(0) = 0,$$

$$\text{B: } \theta(0) = 0, \quad \frac{d\theta}{d\tau}(0) = 2.2.$$

Before integrating, predict from the energy which trajectory librates and which rotates. Then make one phase-plane plot containing both trajectories. Along each numerical solution compute

$$\varepsilon(\tau) = \frac{1}{2} \left(\frac{d\theta}{d\tau} \right)^2 + 1 - \cos \theta$$

and report a simple measure of its numerical variation. Explain whether the computation agrees with the analytical prediction.

Problem 4

Rigid-body kinematics from a reference point matrix. A rigid body consists of material points with masses m_i . Arrange their positions as columns of

$$X(t) = (\mathbf{x}_1(t) \quad \cdots \quad \mathbf{x}_N(t)) \in \mathbb{R}^{3 \times N},$$

and let

$$\mathbf{M} = \text{diag}(m_1, \dots, m_N), \quad \mathbf{1} = (1, \dots, 1)^T, \quad m_T = \mathbf{1}^T \mathbf{M} \mathbf{1}.$$

Choose body coordinates so that the reference configuration X_0 is centered at the center of mass,

$$\boxed{X_0 \mathbf{M} \mathbf{1} = \mathbf{0}.$$

A rigid configuration is written

$$\boxed{X(t) = \mathbf{R}(t) \mathbf{1}^T + Q(t) X_0, \quad Q(t)^T Q(t) = I_3, \quad \det Q(t) = 1.}$$

The matrix Q maps body components into laboratory components.

- (a) Show directly that this map preserves every pairwise distance. Then show that the center of mass of the current configuration is exactly $\mathbf{R}(t)$.
- (b) Define

$$\boxed{W_s \equiv \dot{Q} Q^T.}$$

Differentiate $Q Q^T = I_3$ to show that W_s is skew-symmetric. Explain why in three dimensions there is a unique vector $\boldsymbol{\omega}_s$ such that

$$\boxed{W_s \mathbf{r} = \boldsymbol{\omega}_s \times \mathbf{r}}$$

for every vector $\mathbf{r}$.

- (c) Differentiate the point-matrix configuration and show that

$$\boxed{\dot{X} = \dot{\mathbf{R}} \mathbf{1}^T + W_s (X - \mathbf{R} \mathbf{1}^T).$$

Deduce the velocity of an individual material point,

$$\boxed{\dot{\mathbf{x}}_i = \dot{\mathbf{R}} + \boldsymbol{\omega}_s \times (\mathbf{x}_i - \mathbf{R}).}$$

Use skew symmetry to verify that the instantaneous velocity relative to the center of mass is perpendicular to the radius from the center of mass.

- (d) Consider the explicit orientation

$$Q(t) = Q_z(\Omega t)Q_x(\beta),$$

where Ω and β are constants and

$$Q_x(\beta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \beta & -\sin \beta \\ 0 & \sin \beta & \cos \beta \end{pmatrix}, \quad Q_z(\alpha) = \begin{pmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Calculate $W_s = \dot{Q}Q^T$ and identify $\boldsymbol{\omega}_s$. Then find the same angular velocity in body components,

$$\boldsymbol{\omega}_b = Q^T \boldsymbol{\omega}_s.$$

Explain why $\boldsymbol{\omega}_s$ and $\boldsymbol{\omega}_b$ are not two different physical angular velocities.

- (e) A vector $\mathbf{b}$ has space and body component columns related by

$$\mathbf{b}_s = Q\mathbf{b}_b.$$

Show that

$$Q^T \dot{\mathbf{b}}_s = \dot{\mathbf{b}}_b + \boldsymbol{\omega}_b \times \mathbf{b}_b.$$

Interpret the formula in the two special cases: (i) $\mathbf{b}$ is painted on the body, and (ii) $\mathbf{b}$ is fixed in the laboratory.

Problem 5

Written + Oral Presentation

Rigid-body dynamics: from inertia to torque-free stability. Four equal point masses m form a rigid planar body with body-frame positions

$$(\pm a, 0, 0), \quad (0, \pm b, 0), \quad a > b > 0.$$

The origin is the center of mass and the displayed axes are fixed in the body. For a point mass at $\boldsymbol{\rho}$, its inertia contribution about the origin is

$$m [(\boldsymbol{\rho} \cdot \boldsymbol{\rho})I_3 - \boldsymbol{\rho}\boldsymbol{\rho}^T].$$

- (a) Before carrying out the full sum, use the reflection symmetries of the mass distribution to predict which products of inertia vanish. Then calculate the inertia tensor and identify the principal axes and principal moments. Put the moments in increasing order.
- (b) For angular velocity

$$\boldsymbol{\omega} = \Omega(\mathbf{e}_1 + \mathbf{e}_2),$$

find the body-frame angular momentum $\mathbf{J}$ and rotational kinetic energy. Are $\mathbf{J}$ and $\boldsymbol{\omega}$ parallel? State the condition under which they are parallel for a body with distinct principal moments.

- (c) The laboratory torque and angular momentum satisfy

$$\boldsymbol{\tau}_s = \left(\frac{d\mathbf{J}}{dt} \right)_s.$$

For any vector, the laboratory and body derivatives are related by

$$\left(\frac{d\mathbf{J}}{dt}\right)_s = Q \left[\left(\frac{d\mathbf{J}}{dt}\right)_b + \boldsymbol{\omega}_b \times \mathbf{J}_b \right].$$

Use $\mathbf{J}_b = I\boldsymbol{\omega}_b$ in principal axes to derive Euler's equations

$$I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 = \tau_1,$$

with the other two equations obtained cyclically.

- (d) Set $\boldsymbol{\tau} = 0$ and suppose $I_1 < I_2 < I_3$. Linearize Euler's equations about steady rotation with angular speed Ω about each principal axis. Show that rotations about axes 1 and 3 are linearly stable, while rotation about axis 2 is unstable.
- (e) Now consider a torque-free symmetric top with

$$I_1 = I_2 = I_\perp, \quad I_3 = I_\parallel.$$

Show that ω_3 is constant and that the transverse body components satisfy

$$\dot{\omega}_1 = -\Omega_b\omega_2, \quad \dot{\omega}_2 = \Omega_b\omega_1,$$

where

$$\Omega_b = \frac{I_\parallel - I_\perp}{I_\perp} \omega_3.$$

Interpret the result in both the body frame and the laboratory frame.

- (f) **Oral extension for the designated presenter.** Demonstrate the intermediate-axis instability with a book or rectangular object. Identify which physical axes correspond to the smallest, intermediate, and largest principal moments, and connect the observed behavior to the signs in the linearized equations above. Explain why conservation of laboratory angular momentum does not imply that the body components (J_1, J_2, J_3) remain constant.