

Problem Set 5

PHYS V1100: Analytical Dynamics

Fall 2026

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Due: Monday, October 26

This set develops continuum field mechanics and completes the central-force unit. In several places the physical system is specified but the useful description is not. Choose coordinates, conserved quantities, boundary conditions, or checks that expose the mechanics before beginning detailed algebra. Predictions and independent checks are part of the solution where requested.

Show enough reasoning that another person can follow your argument. Keep boundary terms until their physical meaning is clear, distinguish continuum approximations from separate linearization assumptions, and treat configuration and complete dynamical state as different concepts. Computation is not required for this set. In the written-plus-oral problem, all students complete the written portion; only the designated presenter prepares the oral extension.

Problem 1

A continuum field with an elastic boundary. Consider a one-dimensional continuum field $\phi(x, t)$ on

$$0 \leq x \leq \ell$$

with bulk Lagrangian density

$$\mathcal{L} = \frac{1}{2}\rho\phi_t^2 - \frac{1}{2}\mathcal{T}\phi_x^2.$$

The left endpoint is fixed,

$$\phi(0, t) = 0,$$

while the right endpoint is attached to a linear spring of stiffness k_b . The other end of this spring is fixed, so its extension is $\phi(\ell, t)$.

Variations vanish at the initial and final times:

$$\delta\phi(x, t_1) = \delta\phi(x, t_2) = 0.$$

- (a) **Construct the model before varying it.** Before doing any calculation, predict what boundary behavior you expect in the two limits

$$k_b = 0, \quad k_b \rightarrow \infty.$$

Then construct the complete action for the field plus the endpoint spring. Explain why the spring contribution is a boundary term in the Lagrangian rather than part of the integral over x .

- (b) **Vary the action and keep the boundary term.** Carry out the field variation at fixed spacetime coordinates,

$$\delta x = 0, \quad \delta t = 0.$$

Integrate by parts as needed. Do not discard the spatial boundary contribution before interpreting it.

Derive both the field equation in the interior $0 < x < \ell$ and the boundary condition at $x = \ell$. State clearly why the left boundary contribution vanishes and why the variation at $x = \ell$ is not set to zero.

- (c) **Interpret and check.** Use your boundary condition to recover the two limiting behaviors predicted in part (a).

Then answer briefly:

- (i) Is $\phi(x, t_0)$ by itself a complete dynamical state for this system? If not, what additional initial data are required?
- (ii) In deriving a model of this form from a discrete chain or string, distinguish the continuum/smoothness approximation from a small-slope or linearization approximation. Why are they logically different assumptions?
- (iii) Identify the decisive idea in this problem, one assumption your result uses, and one check that would expose an error.

Problem 2

Reconstruct the discrete mechanics of an elastic boundary. Consider a discrete elastic representation of the same spatial system. Take nodes

$$x_j = ja, \quad j = 0, 1, 2, 3, \quad \ell = 3a,$$

with

$$q_0 = 0$$

at the fixed left boundary. The moving nodal values are

$$\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}.$$

Each of the three bulk segments has effective stiffness

$$\kappa = \frac{\mathcal{T}}{a},$$

and the final node q_3 is attached to the fixed endpoint spring of stiffness k_b .

- (a) **Construct the discrete structure.** Choose an orientation for each elastic element and construct an extension vector

$$\mathbf{e} = A\mathbf{q}.$$

Your model should contain the three bulk elastic segments and the endpoint spring. Construct an appropriate constitutive matrix C and write the elastic potential energy in the form

$$V = \frac{1}{2} \mathbf{e}^T C \mathbf{e}.$$

Different consistent choices of edge orientation are acceptable.

- (b) **From compatibility to force balance.** Starting from your energy, derive the internal elastic force

$$\mathbf{F}_{\text{int}} = -A^T C \mathbf{e} = -A^T C A \mathbf{q}.$$

Compute $A^T C A$. In your explanation, distinguish the roles of A , C , and A^T : what represents compatibility, what represents constitutive response, and what returns edge forces to nodal force balance?

- (c) **Recover the continuum boundary condition.** For this part consider a static configuration, so the last nodal force must balance to zero. Use the last component of $\mathbf{F}_{\text{int}}$ to show that

$$\frac{\mathcal{T}}{a}(q_3 - q_2) + k_b q_3 = 0.$$

Then use

$$q_j \simeq \phi(x_j)$$

to recover the continuum boundary condition from the previous problem.

Finally explain briefly:

- (i) What additional information would be required to turn the vector $\mathbf{q}$ into a complete dynamical state?
- (ii) Does the existence of this discrete model imply that the underlying physical system must be fundamentally discrete?

Problem 3

Written + Oral Presentation

From initial data to a Kepler orbit. The center-of-mass motion of a two-body system has already been separated. Begin with the exact relative-coordinate problem for a particle of reduced mass μ moving in the attractive central potential

$$V(r) = -\frac{k}{r}, \quad k > 0.$$

At $t = 0$, suppose

$$r = r_0, \quad \dot{r} = 0,$$

and the relative velocity is purely tangential with magnitude $v_0 > 0$. Choose the orientation of θ so that the conserved scalar angular momentum is positive.

The written portion is for everyone. The oral extension at the end is prepared only by the student assigned to present this problem in class.

Written portion — all students submit.

- (a) **Choose a useful reduced description and predict the motion.** Choose the conserved quantities or reduced description that you think are most useful for deciding whether the motion is bound and whether the particles can approach $r = 0$. Explain your choice before calculating.

Using the initial data:

- (i) determine the conserved angular momentum J and relative energy E ;
- (ii) construct the effective potential governing the radial motion;
- (iii) explain whether a finite-energy trajectory with $J \neq 0$ can reach $r = 0$;
- (iv) derive the tangential speed v_c that gives a circular orbit at r_0 and the escape speed v_{esc} at r_0 ;
- (v) classify the motion qualitatively for $0 < v_0 < v_{\text{esc}}$, $v_0 = v_{\text{esc}}$, and $v_0 > v_{\text{esc}}$.

For a bound noncircular orbit, also determine whether the initial radius r_0 is periapsis or apoapsis.

- (b) **Recover the orbit shape.** The radial reduction determines $r(t)$, but the actual orbit is a curve in the plane. Eliminate t in favor of θ and choose a variable that makes the orbit equation simple. Derive an equation for the shape $r(\theta)$.

Checkpoint for continuation. Whether or not you complete the reduction, you may use for the remaining work

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu k}{J^2}, \quad u(\theta) = \frac{1}{r(\theta)}.$$

Solve this equation and show that, after choosing the angular origin conveniently,

$$r(\theta) = \frac{p}{1 + e \cos \theta}, \quad p = \frac{J^2}{\mu k}.$$

Explain what information is carried by p and e .

- (c) **Connect the geometry back to the dynamics.** Using a method of your choice, derive or independently verify

$$e^2 = 1 + \frac{2EJ^2}{\mu k^2}.$$

Use this result to connect $E < 0$, $E = 0$, and $E > 0$ with ellipse, parabola, and hyperbola. Finally define

$$\alpha = \frac{\mu r_0 v_0^2}{k} = \frac{v_0^2}{v_c^2}.$$

Show that the initial conditions imply

$$e = |\alpha - 1|,$$

and use this as an independent check of your qualitative predictions in part (a).

Oral extension — designated presenter only. Suppose that near the origin a central potential behaves as

$$V(r) \sim -\frac{a}{r^n}, \quad a > 0.$$

For $J \neq 0$,

$$V_{\text{eff}}(r) \sim -\frac{a}{r^n} + \frac{J^2}{2\mu r^2}.$$

Prepare a short board argument comparing the cases

$$n < 2, \quad n = 2, \quad n > 2.$$

When does the total effective potential necessarily rise to $+\infty$ as $r \rightarrow 0^+$? When can the attractive microscopic potential dominate? What extra comparison is needed in the marginal case $n = 2$?