Problem Set 1

PHYS V1100: Analytical Dynamics

Fall 2026 Professor Mark Shattuck

Due: See Brightspace

Show enough reasoning that another person can follow your argument. Computation may be used where requested, but the purpose is to expose the mechanical structure, not just to obtain a numerical answer.

Problem 1

A useful description of a dynamical system must contain enough information for the dynamics to close. In Lecture 1 we expressed this idea schematically as

$$S_{n+1} = F(S_n).$$

For each situation below, distinguish a proposed state from a reduced or incomplete description.

(a) A one-dimensional particle obeys

$$m\ddot{x} = -kx.$$

Is $x(t)$ alone a complete instantaneous state? Is $(x(t), \dot{x}(t))$ a complete state? Explain.

(b) Two masses are coupled by springs. Suppose the complete mechanical state is

$$S = (x_1, \dot{x}_1, x_2, \dot{x}_2),$$

but an observer records only

$$s = (x_1, \dot{x}_1) = P(S).$$

Why need there not exist an autonomous evolution law $s_{n+1} = f(s_n)$?

(c) Give one way in which the reduced description in part (b) could be made into a better state description without returning to the full microscopic state.

Problem 2

Consider four free masses arranged in a one-dimensional chain between two fixed walls. There are therefore six nodes in all: nodes 0 and 5 are fixed, while nodes 1–4 have displacements

$$\boldsymbol{u} = (u_1, u_2, u_3, u_4)^T.$$

Five springs connect consecutive nodes and have spring constants $c_1, \ldots, c_5 > 0$. Orient every spring from the lower-numbered node toward the higher-numbered node.

(a) Construct the matrix A that maps the four free nodal displacements to the five oriented spring extensions.

Hint. Start from the full incidence matrix for six nodes, then impose $u_0 = u_5 = 0$.

(b) Define

$$C = \mathrm{diag}(c_1, c_2, c_3, c_4, c_5).$$

Write the elastic potential energy in matrix form and explain the role of A and C.

(c) Compute $K = A^T C A$.

(d) Interpret the diagonal and off-diagonal entries of K mechanically. Why is there no rigid-translation zero mode for this system?

(e) If $\boldsymbol{f}$ is the externally applied nodal load, write the static equilibrium equation and identify the internal restoring force.

Problem 3

Consider M free masses between two fixed endpoints. There are $M+1$ identical unit springs, so $C = I$, and each free mass is acted on by the same unit external load,

$$\boldsymbol{f} = \mathbf{1}.$$

Let the endpoint displacements be $u_0 = u_{M+1} = 0$.

(a) Show that the interior equations in

$$A^T A \boldsymbol{u} = \mathbf{1}$$

can be written

$$2u_i - u_{i-1} - u_{i+1} = 1, \qquad i = 1, \ldots, M.$$

(b) Verify directly that

$$u_i = \frac{i(M+1-i)}{2}, \qquad i = 0, 1, \ldots, M+1,$$

satisfies both the boundary conditions and every discrete equilibrium equation.

(c) Reproduce the static solve in either MATLAB or Python/NumPy for $M = 30$. Plot the computed nodal displacements together with the quadratic expression from part (b). Your code should form A, C, $\boldsymbol{f}$, and $A^T C A$ explicitly.

(d) Copy your commands into a script. Change exactly one of the following and briefly describe what changes physically and in the solution:

(i) the number of free masses M;

(ii) the diagonal entries of C;

(iii) the applied load $\boldsymbol{f}$.

Problem 4

Oral Presentation

Let $A \in \mathbb{R}^{N \times (N+1)}$ be the full incidence matrix of an open chain with nodes $0, 1, \ldots, N$, with every edge oriented from left to right. Thus row i of A computes $u_{i+1} - u_i$. Let $\mathbf{1}_E$ be the vector of ones in edge space, and let $\boldsymbol{e}^{(j)}$ be the nodal basis vector with components $(\boldsymbol{e}^{(j)})_i = \delta_{ij}$.

(a) For a chain with four edges, write A explicitly and calculate $A^T \mathbf{1}_E$.

(b) Generalize the result to N edges and show that

$$A^T \mathbf{1}_E = -\boldsymbol{e}^{(0)} + \boldsymbol{e}^{(N)}.$$

What does the support of this vector tell you?

(c) Use only matrix identities to show

$$\mathbf{1}_E^T A\boldsymbol{u} = u_N - u_0.$$

Then write the same identity as an explicit telescoping sum.

(d) Explain in words why this is a discrete fundamental theorem of calculus. Compare it with

$$\int_a^b u_x\,dx = u(b) - u(a).$$

(e) Now replace the open chain by a ring with consistently oriented edges. What do you expect for $A_{\text{ring}}^T \mathbf{1}_E$, and why?

Problem 5

Use the same open-chain incidence matrix A as in the discrete fundamental theorem problem. Let $\boldsymbol{v} \in \mathbb{R}^N$ be an arbitrary quantity living on the edges, with components $v_0, \ldots, v_{N-1}$.

(a) Explain why the scalar identity

$$\boldsymbol{v}^T A\boldsymbol{u} = \boldsymbol{u}^T A^T \boldsymbol{v} = (A^T\boldsymbol{v})^T \boldsymbol{u}$$

is true without making any reference to calculus.

(b) Calculate the components of $A^T\boldsymbol{v}$. Separate the two boundary components from the interior components.

(c) Expand $\boldsymbol{v}^T A\boldsymbol{u} = \boldsymbol{u}^T A^T \boldsymbol{v}$ in components and show that

$$\sum_{i=0}^{N-1} v_i(u_{i+1} - u_i) = -\sum_{i=1}^{N-1} u_i(v_i - v_{i-1}) + u_N v_{N-1} - u_0 v_0.$$

(d) Compare the discrete identity with

$$\int_a^b v u_x\,dx = [vu]_a^b - \int_a^b v_x u\,dx.$$

Which part of $A^T\boldsymbol{v}$ acts like an interior derivative, and which part carries the boundary information?

(e) Recover the discrete fundamental theorem of calculus as a special case by choosing an appropriate $\boldsymbol{v}$.

Problem 6

Challenge

Let A be the incidence matrix of an oriented graph. An edge-space vector $\boldsymbol{c}$ is used to describe an oriented path: its component is $+1$ when the path uses an edge in the chosen edge orientation, -1 when it uses the edge in the opposite direction, and 0 when it does not use that edge.

(a) Consider a simple path beginning at node a and ending at node b. Argue that all interior node contributions cancel in $A^T\boldsymbol{c}$, and show that

$$A^T\boldsymbol{c} = -\boldsymbol{e}^{(a)} + \boldsymbol{e}^{(b)}.$$

(b) Use part (a) to show

$$\boldsymbol{c}^T A\boldsymbol{u} = u_b - u_a.$$

Interpret the result.

(c) What changes if $\boldsymbol{c}$ describes a closed cycle?

(d) Relate the result schematically to the idea

$$\partial^2 = 0,$$

"the boundary of a boundary is zero." No formal algebraic-topology machinery is required.