Graduate Classical Mechanics
Lecture 12: Small Oscillations II and Beyond Linearization

Mass-weighted modes, matrix evolution, zero modes, and nonlinear phase space

1 The organizing picture

Lecture 11 ended with the general linearized small-oscillation problem

$$\boxed{M\ddot{\boldsymbol{\xi}} + K\boldsymbol{\xi} = \mathbf{0},}$$

and the generalized normal-mode eigenvalue problem

$$\boxed{K\boldsymbol{u}_\alpha = \omega_\alpha^2 M\boldsymbol{u}_\alpha.}$$

Here $M = M^T > 0$ is the mass matrix and $K = K^T$ is the quadratic stiffness matrix evaluated at the equilibrium.

The purpose of this lecture is to turn that eigenvalue problem into a complete coordinate transformation. We will carry both the component and matrix descriptions in parallel:

$$\boxed{\begin{array}{ccc} D\mathbf{v}_\alpha = \omega_\alpha^2 \mathbf{v}_\alpha & \Longleftrightarrow & DV = V\Omega^2 \\ \Downarrow & & \Downarrow \\ \boldsymbol{u}_\alpha = M^{-1/2}\mathbf{v}_\alpha & \Longleftrightarrow & U = M^{-1/2}V \\ \Downarrow & & \Downarrow \\ \boldsymbol{\xi} = \displaystyle\sum_\alpha \boldsymbol{u}_\alpha a_\alpha & \Longleftrightarrow & \boldsymbol{\xi} = U\boldsymbol{a} \\ \Downarrow & & \Downarrow \\ \ddot{a}_\alpha + \omega_\alpha^2 a_\alpha = 0 & \Longleftrightarrow & \ddot{\boldsymbol{a}} + \Omega^2\boldsymbol{a} = \mathbf{0}. \end{array}}$$

Once the quadratic dynamics have been separated, we will then ask what the approximation has omitted: mode coupling, amplitude dependence, separatrices, and stability questions that cannot be decided from the quadratic form alone.

2 Mass weighting converts the generalized eigenproblem into a symmetric one

The generalized eigenvalue problem

$$K\boldsymbol{u}_\alpha = \omega_\alpha^2 M\boldsymbol{u}_\alpha$$

contains two symmetric matrices rather than one. Because M is positive definite, it has a unique positive-definite square root $M^{1/2}$. Define the mass-weighted displacement

$$\boxed{\boldsymbol{y} = M^{1/2}\boldsymbol{\xi}.}$$

Then

$$\boldsymbol{\xi} = M^{-1/2}\boldsymbol{y}, \qquad \ddot{\boldsymbol{\xi}} = M^{-1/2}\ddot{\boldsymbol{y}}.$$

Substituting into the equations of motion gives

$$MM^{-1/2}\ddot{\boldsymbol{y}} + KM^{-1/2}\boldsymbol{y} = \mathbf{0}.$$

Multiplying by $M^{-1/2}$ from the left,

$$\boxed{\ddot{\boldsymbol{y}} + D\boldsymbol{y} = \mathbf{0}, \qquad D \equiv M^{-1/2}KM^{-1/2}.}$$

Since M and K are symmetric,

$$\boxed{D = D^T.}$$

Thus the generalized small-oscillation problem has been converted into an ordinary symmetric eigenvalue problem.

Choose a complete orthonormal eigenbasis of D:

$$\boxed{D\mathbf{v}_\alpha = \omega_\alpha^2 \mathbf{v}_\alpha.}$$

Collect the eigenvectors as columns of the matrix

$$\boxed{V = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_N \end{pmatrix}, \qquad V^T V = I.}$$

Define

$$\boxed{\Omega^2 = \mathrm{diag}(\omega_1^2, \ldots, \omega_N^2).}$$

Then all of the component eigenvalue equations are summarized by

$$\boxed{DV = V\Omega^2.}$$

The order matters: the $\mathbf{v}_\alpha$ are the *columns* of V, so the eigenvalues multiply from the right.

For the nonnegative part of the spectrum we choose $\omega_\alpha \geq 0$ and write

$$\boxed{\Omega = \mathrm{diag}(\omega_1, \ldots, \omega_N).}$$

Negative values of ω_α^2 will be interpreted later as linear instability.

3 Return to the original coordinates: the modal matrix U

The mass-weighted eigenvectors $\mathbf{v}_\alpha$ live in the Euclidean $\boldsymbol{y}$-coordinates. Their corresponding mode shapes in the original generalized coordinates are

$$\boxed{\boldsymbol{u}_\alpha = M^{-1/2}\mathbf{v}_\alpha.}$$

Collecting them as columns gives

$$\boxed{U = \begin{pmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 & \cdots & \boldsymbol{u}_N \end{pmatrix} = M^{-1/2}V.}$$

Starting from $DV = V\Omega^2$,

$$M^{-1/2}KM^{-1/2}V = V\Omega^2.$$

Multiplying by $M^{1/2}$,

$$K(M^{-1/2}V) = M(M^{-1/2}V)\Omega^2.$$

Therefore

$$\boxed{KU = MU\Omega^2,}$$

which is the matrix form of

$$\boxed{K\boldsymbol{u}_\alpha = \omega_\alpha^2 M\boldsymbol{u}_\alpha.}$$

This construction also makes the natural orthogonality immediate:

$$\begin{aligned} U^T MU &= V^T M^{-1/2} M M^{-1/2} V \\ &= V^T V \\ &= I. \end{aligned}$$

Hence

$$\boxed{U^T MU = I} \qquad \Longleftrightarrow \qquad \boxed{\boldsymbol{u}_\alpha^T M\boldsymbol{u}_\beta = \delta_{\alpha\beta}.}$$

The normal modes are therefore orthonormal in the M-weighted inner product

$$\boxed{\langle \boldsymbol{u}_\alpha, \boldsymbol{u}_\beta \rangle_M \equiv \boldsymbol{u}_\alpha^T M\boldsymbol{u}_\beta.}$$

Similarly, using $KU = MU\Omega^2$,

$$U^T KU = U^T MU\Omega^2 = \Omega^2,$$

so

$$\boxed{U^T KU = \Omega^2} \qquad \Longleftrightarrow \qquad \boxed{\boldsymbol{u}_\alpha^T K\boldsymbol{u}_\beta = \omega_\alpha^2 \delta_{\alpha\beta}.}$$

Thus the same modal basis simultaneously reduces the kinetic and potential quadratic forms.

3.1 Direct component proof of mass orthogonality

The matrix construction is efficient, but the component proof is also useful. Take two modes:

$$K\boldsymbol{u}_\alpha = \omega_\alpha^2 M\boldsymbol{u}_\alpha, \qquad K\boldsymbol{u}_\beta = \omega_\beta^2 M\boldsymbol{u}_\beta.$$

Left-multiply the first by $\boldsymbol{u}_\beta^T$ and the second by $\boldsymbol{u}_\alpha^T$. Since M and K are symmetric,

$$\boxed{(\omega_\alpha^2 - \omega_\beta^2)\boldsymbol{u}_\beta^T M\boldsymbol{u}_\alpha = 0.}$$

For distinct eigenvalues,

$$\boxed{\omega_\alpha^2 \neq \omega_\beta^2 \quad \Longrightarrow \quad \boldsymbol{u}_\beta^T M\boldsymbol{u}_\alpha = 0.}$$

If the eigenvalues are degenerate, the equation imposes no orthogonality condition by itself. Because D is symmetric, however, we may choose an orthonormal basis inside the degenerate eigenspace, which induces an M-orthonormal basis in the original coordinates.

4 Normal coordinates diagonalize the dynamics

Expand the displacement in the mode basis:

$$\boxed{\boldsymbol{\xi} = U\boldsymbol{a}} \qquad \Longleftrightarrow \qquad \boxed{\boldsymbol{\xi} = \sum_{\alpha=1}^{N} \boldsymbol{u}_\alpha a_\alpha.}$$

Here

$$\boldsymbol{a} = (a_1, \ldots, a_N)^T$$

is the vector of normal coordinates.

Substitute $\boldsymbol{\xi} = U\boldsymbol{a}$ into the linearized equations:

$$MU\ddot{\boldsymbol{a}} + KU\boldsymbol{a} = \boldsymbol{0}.$$

Left-multiply by U^T:

$$U^T MU\ddot{\boldsymbol{a}} + U^T KU\boldsymbol{a} = \boldsymbol{0}.$$

Using

$$U^T MU = I, \qquad U^T KU = \Omega^2,$$

we obtain

$$\boxed{\ddot{\boldsymbol{a}} + \Omega^2 \boldsymbol{a} = \boldsymbol{0}.}$$

Componentwise this is simply

$$\boxed{\ddot{a}_\alpha + \omega_\alpha^2 a_\alpha = 0.}$$

Thus the original coupled system has become a set of independent scalar equations.

The quadratic kinetic energy becomes

$$\begin{aligned} T^{(2)} &= \frac{1}{2}\dot{\boldsymbol{\xi}}^T M\dot{\boldsymbol{\xi}} \\ &= \frac{1}{2}\dot{\boldsymbol{a}}^T U^T MU\dot{\boldsymbol{a}} \\ &= \boxed{\frac{1}{2}\dot{\boldsymbol{a}}^T\dot{\boldsymbol{a}}} = \boxed{\sum_{\alpha=1}^{N} \frac{1}{2}\dot{a}_\alpha^2}. \end{aligned}$$

The quadratic potential energy becomes

$$\begin{aligned} V^{(2)} &= \frac{1}{2}\boldsymbol{\xi}^T K\boldsymbol{\xi} \\ &= \frac{1}{2}\boldsymbol{a}^T U^T KU\boldsymbol{a} \\ &= \boxed{\frac{1}{2}\boldsymbol{a}^T\Omega^2\boldsymbol{a}} = \boxed{\sum_{\alpha=1}^{N} \frac{1}{2}\omega_\alpha^2 a_\alpha^2}. \end{aligned}$$

Hence

$$\boxed{L^{(2)} = \frac{1}{2}\dot{\boldsymbol{a}}^T\dot{\boldsymbol{a}} - \frac{1}{2}\boldsymbol{a}^T\Omega^2\boldsymbol{a} = \sum_{\alpha=1}^{N}\left[\frac{1}{2}\dot{a}_\alpha^2 - \frac{1}{2}\omega_\alpha^2 a_\alpha^2\right].}$$

The generalized eigenproblem therefore does more than find frequencies: it constructs coordinates in which both quadratic forms become diagonal.

5 Projecting the initial state into modal coordinates

Because

$$\boldsymbol{\xi} = U\boldsymbol{a}$$

and

$$U^T M U = I,$$

we may recover the modal amplitudes directly:

$$\boxed{\boldsymbol{a} = U^T M \boldsymbol{\xi}.}$$

Componentwise,

$$\boxed{a_\alpha = \boldsymbol{u}_\alpha^T M \boldsymbol{\xi}.}$$

Therefore the complete initial state projects as

$$\boxed{\boldsymbol{a}(0) = U^T M \boldsymbol{\xi}(0), \qquad \dot{\boldsymbol{a}}(0) = U^T M \dot{\boldsymbol{\xi}}(0),}$$

or, for each mode,

$$\boxed{a_\alpha(0) = \boldsymbol{u}_\alpha^T M \boldsymbol{\xi}(0), \qquad \dot{a}_\alpha(0) = \boldsymbol{u}_\alpha^T M \dot{\boldsymbol{\xi}}(0).}$$

A mode decomposition acts on the full dynamical state: both the initial displacement and the initial velocity are required.

6 Positive-frequency evolution: component and matrix forms

For a mode with $\omega_\alpha > 0$, the scalar initial-value problem

$$\ddot{a}_\alpha + \omega_\alpha^2 a_\alpha = 0$$

has the solution

$$\boxed{a_\alpha(t) = a_\alpha(0)\cos(\omega_\alpha t) + \frac{\dot{a}_\alpha(0)}{\omega_\alpha}\sin(\omega_\alpha t).}$$

If every mode in the block has strictly positive frequency, the same result is written compactly as

$$\boxed{\boldsymbol{a}(t) = \cos(\Omega t)\boldsymbol{a}(0) + \Omega^{-1}\sin(\Omega t)\dot{\boldsymbol{a}}(0).}$$

Since Ω is diagonal, the matrix functions are simply

$$\cos(\Omega t) = \operatorname{diag}\big(\cos\omega_1 t, \ldots, \cos\omega_N t\big),$$

and

$$\Omega^{-1}\sin(\Omega t) = \operatorname{diag}\left(\frac{\sin\omega_1 t}{\omega_1}, \ldots, \frac{\sin\omega_N t}{\omega_N}\right).$$

The initial conditions are built into this form. At $t = 0$,

$$\cos(0) = I, \qquad \sin(0) = 0,$$

so

$$\boldsymbol{a}(0) = \boldsymbol{a}(0).$$

Differentiating,

$$\dot{\boldsymbol{a}}(t) = -\Omega \sin(\Omega t)\boldsymbol{a}(0) + \cos(\Omega t)\dot{\boldsymbol{a}}(0),$$

which gives the correct $\dot{\boldsymbol{a}}(0)$. Differentiating again yields

$$\ddot{\boldsymbol{a}}(t) = -\Omega^2 \boldsymbol{a}(t).$$

7 Zero modes and the null-space-resolved propagator

A zero-frequency mode must be treated differently. If

$$\omega_0 = 0,$$

then

$$\ddot{a}_0 = 0,$$

so

$$\boxed{a_0(t) = a_0(0) + \dot{a}_0(0)t.}$$

A zero-frequency mode is therefore not necessarily static; it may describe uniform drift along a flat or symmetry direction.

For a nonnegative spectrum, order the columns of U so that the positive-frequency modes appear first and the zero modes last:

$$\boxed{U = \begin{pmatrix} U_+ & U_0 \end{pmatrix}, \qquad \Omega = \begin{pmatrix} \Omega_+ & 0 \\ 0 & 0 \end{pmatrix}, \qquad \boldsymbol{a} = \begin{pmatrix} \boldsymbol{a}_+ \\ \boldsymbol{a}_0 \end{pmatrix}, \qquad \Omega_+ > 0.}$$

Here U_0 spans the zero-mode subspace. The complete modal evolution can then be written without any singular inverse:

$$\boxed{\boldsymbol{a}(t) = \begin{pmatrix} \cos(\Omega_+ t) & 0 \\ 0 & I_0 \end{pmatrix} \boldsymbol{a}(0) + \begin{pmatrix} \Omega_+^{-1} \sin(\Omega_+ t) & 0 \\ 0 & tI_0 \end{pmatrix} \dot{\boldsymbol{a}}(0).}$$

The upper block is the usual oscillatory propagator. The lower block is free motion in the null space.

Together with

$$\boxed{\boldsymbol{a}(0) = U^T M \boldsymbol{\xi}(0), \qquad \dot{\boldsymbol{a}}(0) = U^T M \dot{\boldsymbol{\xi}}(0), \qquad \boldsymbol{\xi}(t) = U\boldsymbol{a}(t),}$$

this gives the complete solution in the original generalized coordinates.

The same result in explicit indexed form is

$$\boxed{\begin{aligned} \boldsymbol{\xi}(t) = &\sum_{\omega_\alpha > 0} \boldsymbol{u}_\alpha \left[a_\alpha(0)\cos(\omega_\alpha t) + \frac{\dot{a}_\alpha(0)}{\omega_\alpha} \sin(\omega_\alpha t) \right] \\ &+ \sum_{\omega_\alpha = 0} \boldsymbol{u}_\alpha \left[a_\alpha(0) + \dot{a}_\alpha(0) t \right]. \end{aligned}}$$

Optional computational viewpoint

The zero-mode block can also be written using a Moore–Penrose pseudoinverse together with the projector onto the null space, or by using the analytic matrix function

$$t\,\operatorname{sinc}(\Omega t), \qquad \operatorname{sinc} x = \frac{\sin x}{x}, \qquad \operatorname{sinc} 0 = 1.$$

Those forms are useful computationally and conceptually, but the explicit block form already contains the essential structure without requiring additional machinery.

8 Degeneracy and basis freedom

Degeneracy and zero frequency are different ideas. A degenerate eigenspace contains several independent directions with the same eigenvalue. A zero mode has eigenvalue $\omega^2 = 0$. A zero eigenspace may itself be degenerate, but degeneracy does not imply zero frequency.

The simplest example is an isotropic two-dimensional oscillator:

$$\boxed{M = mI, \qquad K = kI.}$$

Then every nonzero vector $\boldsymbol{u}$ satisfies

$$K\boldsymbol{u} = \frac{k}{m} M\boldsymbol{u},$$

so every direction has

$$\boxed{\omega^2 = \frac{k}{m}.}$$

There is no unique preferred pair of modal directions. Any rotated orthonormal basis spans the same eigenspace.

In matrix language, suppose U_d contains an M-orthonormal basis for a degenerate block with frequency ω. For any orthogonal matrix R,

$$\boxed{U_d \longrightarrow U_d R, \qquad R^T R = I.}$$

Then

$$\boxed{(U_d R)^T M (U_d R) = I,}$$

and

$$\boxed{(U_d R)^T K (U_d R) = \omega^2 I.}$$

The individual basis vectors change, but the degenerate subspace and its reduced dynamics do not.

This is an important general lesson: the eigenspace is physical; a particular orthonormal basis inside a degenerate eigenspace is a choice.

9 Continuum version: weighted orthogonality becomes an integral

The finite-dimensional modal inner product is

$$\boxed{\langle \boldsymbol{u}_\alpha, \boldsymbol{u}_\beta \rangle_M = \boldsymbol{u}_\alpha^T M \boldsymbol{u}_\beta.}$$

In a continuum system, the mass matrix is replaced by a mass density or weight function. For a one-dimensional field with density $\rho(x)$, normalized modes satisfy

$$\boxed{\int_0^\ell \rho(x) u_\alpha(x) u_\beta(x)\, dx = \delta_{\alpha\beta}.}$$

A discrete modal expansion

$$\boldsymbol{\xi}(t) = \sum_\alpha \boldsymbol{u}_\alpha a_\alpha(t)$$

becomes the continuum spectral expansion

$$\boxed{\phi(x,t) = \sum_\alpha a_\alpha(t) u_\alpha(x).}$$

Thus finite-dimensional normal modes and continuum mode expansions are the same structural move: expand the configuration in a basis adapted to the dynamical operator and its kinetic-energy weight.

10 What the quadratic approximation has discarded

The exact mechanics near an equilibrium contain more than the quadratic Lagrangian

$$L^{(2)} = \frac{1}{2}\dot{\boldsymbol{\xi}}^T M \dot{\boldsymbol{\xi}} - \frac{1}{2}\boldsymbol{\xi}^T K \boldsymbol{\xi}.$$

Schematically,

$$\boxed{L = L^{(2)} + L^{(3)} + L^{(4)} + \cdots.}$$

For the potential,

$$\boxed{V = V_0 + \frac{1}{2}K_{ij}\xi_i\xi_j + \frac{1}{3!}V^{(3)}_{ijk}\xi_i\xi_j\xi_k + \frac{1}{4!}V^{(4)}_{ijkl}\xi_i\xi_j\xi_k\xi_l + \cdots.}$$

In general coordinates, the mass matrix $M(\boldsymbol{q})$ also depends on configuration. Expanding it can generate higher-order kinetic terms such as

$$\xi\,\dot{\xi}^2.$$

These terms were consistently neglected in the quadratic theory.

Transforming to modal coordinates does not make the nonlinear terms disappear. For example, a cubic potential term becomes schematically

$$\boxed{V^{(3)} = \sum_{\alpha\beta\gamma} C_{\alpha\beta\gamma} a_\alpha a_\beta a_\gamma.}$$

Thus

quadratic theory: independent modes and superposition
nonlinear theory: mode coupling and amplitude dependence.

The modal basis remains useful, but the modal amplitudes no longer evolve independently.

11 The nonlinear pendulum as a global phase-space example

The exact pendulum equation is

$$\boxed{\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0.}$$

Its conserved energy is

$$\boxed{E = \frac{1}{2}m\ell^2\dot{\theta}^2 + mg\ell(1-\cos\theta).}$$

Near $\theta = 0$,

$$1 - \cos\theta = \frac{1}{2}\theta^2 - \frac{1}{24}\theta^4 + \cdots,$$

so the quadratic approximation gives

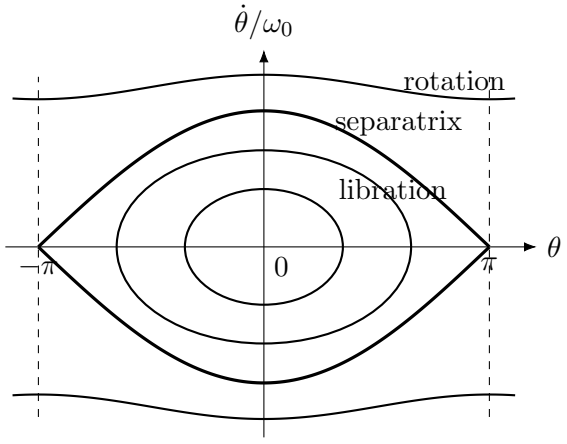
$$\boxed{\ddot{\theta} + \frac{g}{\ell}\theta = 0, \qquad \omega_0 = \sqrt{\frac{g}{\ell}}.}$$

This correctly describes sufficiently small oscillations near the stable equilibrium. It does not describe the global phase portrait.

Solving the exact energy relation for $\dot{\theta}$ gives

$$\boxed{\dot{\theta} = \pm\sqrt{\frac{2}{m\ell^2}\,[E - mg\ell(1-\cos\theta)]}.}$$

These curves organize the phase plane.



The energy classification is

$$\boxed{0 < E < 2mg\ell \quad \Longrightarrow \quad \text{libration: closed phase curves,}}$$

$$\boxed{E = 2mg\ell \quad \Longrightarrow \quad \text{separatrix through } (\pi, 0),}$$

and

$$\boxed{E > 2mg\ell \quad \Longrightarrow \quad \text{continuous rotation.}}$$

The harmonic approximation sees only the local center near $(\theta, \dot{\theta}) = (0, 0)$. It cannot reveal the separatrix or the rotational trajectories because those are global features of the nonlinear system.

The first nonlinear correction to the potential is negative:

$$\boxed{V(\theta) = \frac{1}{2}mg\ell\theta^2 - \frac{1}{24}mg\ell\theta^4 + \cdots .}$$

The pendulum therefore softens as the oscillation amplitude increases; the period is no longer amplitude independent.

12 Marginal stability: when the quadratic theory is inconclusive

Consider

$$\boxed{V_+(x) = \frac{\lambda}{4}x^4, \qquad V_-(x) = -\frac{\lambda}{4}x^4, \qquad \lambda > 0.}$$

For both potentials,

$$\left.\frac{dV}{dx}\right|_{x=0} = 0, \qquad \left.\frac{d^2V}{dx^2}\right|_{x=0} = 0.$$

Thus both have the same quadratic stiffness: zero. Yet their stability is opposite:

$$\boxed{V_+ : \text{ stable minimum}, \qquad V_- : \text{ unstable maximum.}}$$

Higher-order terms decide what the quadratic theory cannot.

For a normal mode,

$$K\boldsymbol{u} = \omega^2 M\boldsymbol{u}.$$

Multiplying from the left by $\boldsymbol{u}^T$,

$$\boxed{\omega^2 = \frac{\boldsymbol{u}^T K\boldsymbol{u}}{\boldsymbol{u}^T M\boldsymbol{u}}.}$$

Because $M > 0$, the sign of ω^2 is controlled by the quadratic restoring form $\boldsymbol{u}^T K\boldsymbol{u}$. This leads to the stability hierarchy

$$\boxed{K > 0 \quad \Longrightarrow \quad \text{strict quadratic minimum and small oscillations,}}$$

$$\boxed{K \text{ has a negative direction} \quad \Longrightarrow \quad \text{some } \omega^2 < 0 \text{ and linear instability,}}$$

and

$$\boxed{K \text{ has a zero direction} \quad \Longrightarrow \quad \text{the quadratic test is inconclusive.}}$$

A zero eigenvalue can therefore have two very different origins:

- an exact continuous symmetry may produce a genuinely flat direction and a true symmetry zero mode;
- a marginal equilibrium may have zero quadratic curvature even though higher-order terms eventually stabilize or destabilize the motion.

13 Summary map

The complete quadratic construction can be summarized as

$$M\ddot{\boldsymbol{\xi}} + K\boldsymbol{\xi} = 0$$

$$\Downarrow \text{ mass weight}$$

$$D = M^{-1/2}KM^{-1/2}, \qquad DV = V\Omega^2$$

$$\Downarrow$$

$$U = M^{-1/2}V, \qquad U^T MU = I, \qquad U^T KU = \Omega^2$$

$$\Downarrow$$

$$\boldsymbol{\xi} = U\boldsymbol{a}, \qquad \ddot{\boldsymbol{a}} + \Omega^2\boldsymbol{a} = 0$$

$$\Downarrow \text{ restore omitted terms}$$

mode coupling, amplitude dependence, separatrices, and global dynamics.

The quadratic theory is powerful because it reduces a coupled many-coordinate system to independent modal coordinates. Its limitations are equally important: zero quadratic curvature may require further analysis, and nonlinear terms restore interactions between the modes and introduce global phase-space structures that no local harmonic approximation can see.