

Graduate Classical Mechanics

Lecture 11: Central Forces II and Small Oscillations I

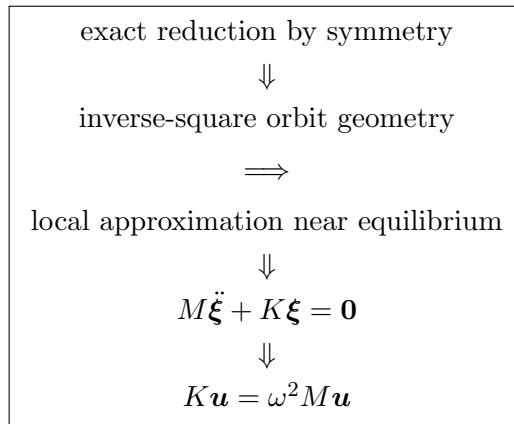
From exact reduction to local quadratic dynamics

1 The organizing transition

Lecture 10 reduced an isolated two-body central-force problem exactly. Translation symmetry separated the center-of-mass motion, rotational symmetry produced conserved angular momentum, and the relative motion became a one-dimensional radial problem plus reconstruction of the angle.

The central-force portion of this lecture finishes that exact reduction by recovering the *shape* of an inverse-square orbit. The second half then changes strategy. A general many-degree-of-freedom system may not possess enough symmetry for an exact reduction, but near an equilibrium its dynamics often simplify to a universal quadratic form.

The lecture therefore has one broad structural arc:



The first simplification is exact and rests on symmetry. The second is approximate and rests on expanding about an equilibrium. Keeping those two ideas distinct will be important later.

I. Central forces II: from radial dynamics to orbit geometry

2 Recall the exact reduction from Lecture 10

After separating the center-of-mass motion, the relative coordinate $\mathbf{r}$ behaves like one particle of reduced mass μ moving in a central potential $V(r)$, with

$$r = \|\mathbf{r}\|.$$

In the orbital plane, polar coordinates give

$$L_{\text{rel}} = \frac{1}{2}\mu \left(\dot{r}^2 + r^2 \dot{\theta}^2 \right) - V(r).$$

The angular coordinate is cyclic, so

$$J = \frac{\partial L_{\text{rel}}}{\partial \dot{\theta}} = \mu r^2 \dot{\theta} = \text{constant}.$$

Because the Lagrangian has no explicit time dependence, the relative energy is also conserved:

$$E = \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\mu r^2 \dot{\theta}^2 + V(r).$$

Eliminating $\dot{\theta}$ with the conserved J gives

$$E = \frac{1}{2}\mu \dot{r}^2 + V_{\text{eff}}(r),$$

where

$$V_{\text{eff}}(r) = V(r) + \frac{J^2}{2\mu r^2}.$$

The second term is angular kinetic energy rewritten as a function of r . It is not a new microscopic interaction.

The radial coordinate therefore obeys the exact one-dimensional equation

$$\mu \ddot{r} = -\frac{dV_{\text{eff}}}{dr}.$$

The angle is reconstructed afterward from

$$\dot{\theta} = \frac{J}{\mu r^2}.$$

The central question for this lecture is now:

The radial reduction tells us how r changes in time. How do we recover the actual curve traced by the orbit in the plane?

3 Inverse-square attraction

Take the attractive inverse-square potential

$$V(r) = -\frac{k}{r}, \quad k > 0.$$

Then

$$V_{\text{eff}}(r) = -\frac{k}{r} + \frac{J^2}{2\mu r^2}.$$

For $J \neq 0$, the two limiting behaviors are

$$V_{\text{eff}}(r) \rightarrow +\infty \quad (r \rightarrow 0^+),$$

because the positive $1/r^2$ term dominates the attractive $1/r$ term, and

$$V_{\text{eff}}(r) \rightarrow 0^- \quad (r \rightarrow \infty).$$

Thus the effective potential rises steeply at small radius, has a negative minimum, and approaches zero from below at large radius.

3.1 A dimensionless schematic

It is useful to scale out the parameters before drawing the shape. Define the circular-orbit radius

$$r_c = \frac{J^2}{\mu k}$$

and the dimensionless radius

$$x = \frac{r}{r_c}.$$

If we measure energy in units

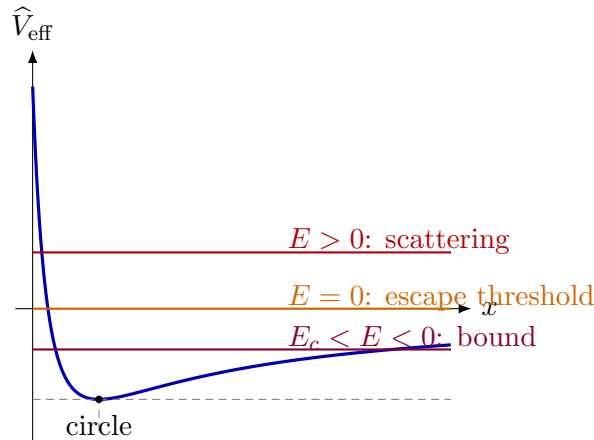
$$E_0 = \frac{\mu k^2}{2J^2},$$

then

$$\hat{V}_{\text{eff}}(x) \equiv \frac{V_{\text{eff}}}{E_0} = \frac{1}{x^2} - \frac{2}{x}.$$

The minimum occurs at

$$x = 1, \quad \hat{V}_{\text{eff}} = -1.$$



This one picture already organizes the possible radial behaviors.

3.2 Allowed regions and turning points

Because

$$\frac{1}{2}\mu\dot{r}^2 = E - V_{\text{eff}}(r),$$

we must have

$$V_{\text{eff}}(r) \leq E.$$

A turning point occurs when

$$\dot{r} = 0 \iff V_{\text{eff}}(r) = E.$$

The horizontal energy line therefore gives the allowed range of r immediately.

For the minimum energy,

$$E = E_c,$$

the energy line only touches the effective potential at one radius. The radial coordinate is constant and the orbit is circular.

For

$$E_c < E < 0,$$

the energy line intersects V_{eff} twice. The radial coordinate is trapped between two finite turning radii:

$$r_{\min} \leq r \leq r_{\max}.$$

The orbit is bound.

For

$$E = 0,$$

the particle has a finite inner turning point but may reach arbitrarily large r with asymptotically vanishing radial speed. This is the threshold between bound and unbound motion.

For

$$E > 0,$$

there is one finite inner turning point and the allowed region extends to $r \rightarrow \infty$. A particle can come in from far away, reach a closest approach, and return to infinity. This is scattering.

Thus, before solving any orbit equation, the effective-potential picture already distinguishes

$$\boxed{\text{circular} \quad \text{vs.} \quad \text{bound} \quad \text{vs.} \quad \text{escape threshold} \quad \text{vs.} \quad \text{scattering.}}$$

4 Circular orbit and radial stability

A circular orbit occurs at an extremum of the effective potential:

$$\frac{dV_{\text{eff}}}{dr} = \frac{k}{r^2} - \frac{J^2}{\mu r^3} = 0.$$

Hence

$$\boxed{r_c = \frac{J^2}{\mu k}}.$$

Its energy is the minimum value of V_{eff} :

$$E_c = V_{\text{eff}}(r_c) = -\frac{\mu k^2}{2J^2}.$$

The second derivative is

$$\frac{d^2 V_{\text{eff}}}{dr^2} = -\frac{2k}{r^3} + \frac{3J^2}{\mu r^4}.$$

At $r = r_c$, using

$$\frac{J^2}{\mu} = k r_c,$$

we obtain

$$\left. \frac{d^2 V_{\text{eff}}}{dr^2} \right|_{r_c} = \frac{k}{r_c^3} > 0.$$

The circular orbit is therefore radially stable.

This statement is local: sufficiently small radial perturbations oscillate about the minimum. It does not by itself determine every aspect of the full orbit geometry. For that we now eliminate time.

5 Orbit geometry: eliminate time instead of solving for $r(t)$

An orbit is the geometric relation between r and θ , so solving first for $r(t)$ and then for $\theta(t)$ is not always the most direct route.

Angular-momentum conservation gives

$$\dot{\theta} = \frac{J}{\mu r^2}.$$

Introduce

$$u(\theta) = \frac{1}{r(\theta)}.$$

Then

$$r = \frac{1}{u},$$

and

$$\frac{dr}{d\theta} = -\frac{1}{u^2} \frac{du}{d\theta}.$$

Therefore

$$\begin{aligned} \dot{r} &= \frac{dr}{d\theta} \dot{\theta} \\ &= -\frac{1}{u^2} \frac{du}{d\theta} \frac{J}{\mu} u^2 \\ &= -\frac{J}{\mu} \frac{du}{d\theta}. \end{aligned}$$

Differentiate once more:

$$\begin{aligned}\ddot{r} &= -\frac{J}{\mu} \frac{d}{dt} \left(\frac{du}{d\theta} \right) \\ &= -\frac{J}{\mu} \frac{d^2 u}{d\theta^2} \dot{\theta} \\ &= \boxed{-\frac{J^2}{\mu^2} u^2 \frac{d^2 u}{d\theta^2}}.\end{aligned}$$

Also,

$$\boxed{r\dot{\theta}^2 = \frac{J^2}{\mu^2} u^3}.$$

The radial component of acceleration is

$$\ddot{r} - r\dot{\theta}^2,$$

so

$$\boxed{\ddot{r} - r\dot{\theta}^2 = -\frac{J^2}{\mu^2} u^2 \left(\frac{d^2 u}{d\theta^2} + u \right)}.$$

For the inverse-square force,

$$\mu(\ddot{r} - r\dot{\theta}^2) = -\frac{k}{r^2} = -ku^2.$$

Therefore

$$-\frac{J^2}{\mu} u^2 \left(\frac{d^2 u}{d\theta^2} + u \right) = -ku^2.$$

For finite r , divide by $-u^2$:

$$\boxed{\frac{d^2 u}{d\theta^2} + u = \frac{\mu k}{J^2}}.$$

This is the Binet equation specialized to an inverse-square attraction. A striking simplification has occurred: the orbit problem has become a shifted harmonic-oscillator equation, with θ rather than t as the independent variable.

6 The conic-section orbit

The solution of

$$\frac{d^2 u}{d\theta^2} + u = \frac{\mu k}{J^2}$$

is

$$u(\theta) = \frac{\mu k}{J^2} + A \cos \theta + B \sin \theta.$$

The oscillatory part can be written as a shifted cosine. Choosing the angular origin along an apsidal direction gives

$$\boxed{u(\theta) = \frac{\mu k}{J^2} (1 + e \cos \theta)},$$

where $e \geq 0$ is the eccentricity.

Since $u = 1/r$,

$$r(\theta) = \frac{p}{1 + e \cos \theta},$$

with

$$p = \frac{J^2}{\mu k}.$$

The parameter p sets the orbital length scale and is determined by angular momentum. The eccentricity e determines the conic shape:

$$\begin{array}{ll} 0 \leq e < 1 & : \text{ ellipse,} \\ e = 1 & : \text{ parabola,} \\ e > 1 & : \text{ hyperbola.} \end{array}$$

The circular case is

$$e = 0,$$

which gives

$$r = p = \frac{J^2}{\mu k} = r_c.$$

Thus the circle derived from the orbit equation agrees with the minimum of V_{eff} .

7 Energy determines the conic class

The orbit geometry should reproduce the bound/unbound classification already read from the effective potential.

From

$$\dot{r} = -\frac{J}{\mu} \frac{du}{d\theta},$$

the energy can be written as

$$E = \frac{J^2}{2\mu} \left[\left(\frac{du}{d\theta} \right)^2 + u^2 \right] - ku.$$

Insert

$$u = \frac{\mu k}{J^2} (1 + e \cos \theta),$$

so

$$\frac{du}{d\theta} = -\frac{\mu k}{J^2} e \sin \theta.$$

Then

$$\left(\frac{du}{d\theta} \right)^2 + u^2 = \left(\frac{\mu k}{J^2} \right)^2 (1 + 2e \cos \theta + e^2).$$

Substitution into the energy cancels the θ -dependent terms, leaving

$$E = \frac{\mu k^2}{2J^2} (e^2 - 1).$$

Therefore

$$e^2 = 1 + \frac{2EJ^2}{\mu k^2}.$$

The correspondence is immediate:

$E < 0$	$0 \leq e < 1$	bound ellipse
$E = 0$	$e = 1$	parabolic escape threshold
$E > 0$	$e > 1$	hyperbolic scattering

This is an important validation: the effective-potential picture and the orbit geometry independently classify the motion in the same way.

8 Scattering as the unbound branch

For $E > 0$, the orbit is hyperbolic and extends to infinity. The particle approaches from large distance, reaches a finite closest approach, and returns to infinity.

At large r ,

$$u = \frac{1}{r} \rightarrow 0,$$

so the asymptotic directions satisfy

$$1 + e \cos \theta_\infty = 0.$$

Hence

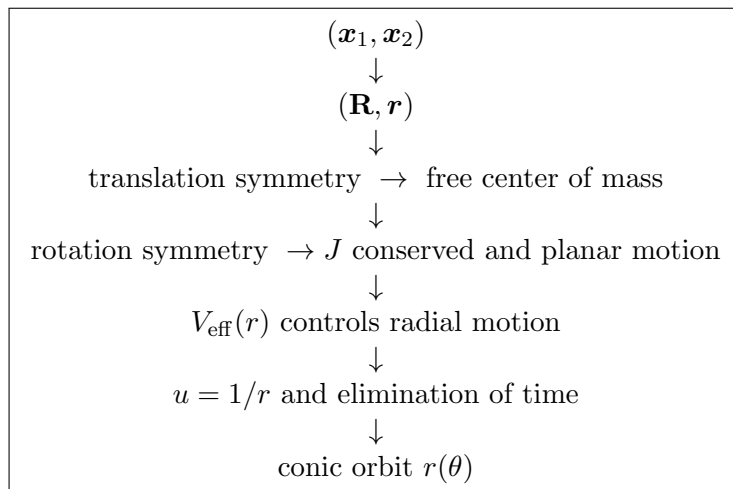
$$\cos \theta_\infty = -\frac{1}{e}.$$

The incoming and outgoing asymptotes determine a scattering deflection angle. A detailed impact-parameter formula or differential cross section is not needed here. The structural point is simpler:

$$E > 0 \implies \text{hyperbolic orbit} \implies \text{scattering from infinity to infinity.}$$

9 What the central-force unit has accomplished

The full reduction is now



This is an exact reduction. No approximation was needed to arrive at the conic orbit for the inverse-square potential.

II. Small oscillations I: local quadratic dynamics

10 A different kind of simplification

Suppose now that a mechanical system has many generalized coordinates and no symmetry strong enough to reduce it exactly. There is still a broadly useful question:

What does the system look like sufficiently close to a stable equilibrium?

The strategy is no longer exact symmetry reduction. Instead we expand the mechanics locally about an equilibrium and keep the leading nontrivial terms.

The transition is therefore

exact reduction by symmetry → local quadratic approximation near equilibrium.

The word *small* in small oscillations refers to the amplitude of the displacement from equilibrium, not to a small number of degrees of freedom.

11 Equilibrium and the quadratic potential

Let the generalized coordinates be

$$\mathbf{q} = (q_1, \dots, q_N)^T.$$

Assume a conservative, time-independent system with potential energy

$$V = V(\mathbf{q}).$$

A static equilibrium $\mathbf{q}^{(0)}$ satisfies

$$\left. \frac{\partial V}{\partial \mathbf{q}} \right|_{\mathbf{q}^{(0)}} = \mathbf{0}.$$

Define the displacement from equilibrium

$$\boldsymbol{\xi} = \mathbf{q} - \mathbf{q}^{(0)}.$$

Taylor-expand the potential:

$$\begin{aligned} V(\mathbf{q}^{(0)} + \boldsymbol{\xi}) &= V(\mathbf{q}^{(0)}) + \left. \frac{\partial V}{\partial \mathbf{q}} \right|_0^T \boldsymbol{\xi} \\ &\quad + \frac{1}{2} \sum_{i,j} K_{ij} \xi_i \xi_j + O(\xi^3), \end{aligned}$$

where

$$K_{ij} = \left. \frac{\partial^2 V}{\partial q_i \partial q_j} \right|_{\mathbf{q}^{(0)}}.$$

The linear term vanishes at equilibrium. Dropping the irrelevant constant $V(\mathbf{q}^{(0)})$, the leading potential energy is

$$V^{(2)} = \frac{1}{2} \boldsymbol{\xi}^T K \boldsymbol{\xi}.$$

The matrix K is the Hessian of the potential at the equilibrium. It is the general local stiffness matrix.

11.1 Stability information in K

If K is positive definite, then the equilibrium is a strict local minimum of the quadratic approximation and small displacements increase the potential energy in every direction.

If K has a negative direction, then moving in that direction lowers the potential to quadratic order and the equilibrium is unstable.

If K has a zero eigenvalue, the quadratic test is not enough by itself. A zero direction may come from an exact continuous symmetry, in which case the potential is genuinely flat along that symmetry direction, or it may require cubic and quartic terms to decide stability.

Thus

$$\lambda_K > 0 \text{ in all directions} \Rightarrow \text{quadratic stability,}$$

while

$$\lambda_K = 0 \Rightarrow \text{inspect symmetry or higher-order terms.}$$

12 How this K connects to the earlier node-edge structure

For a spring network the elastic energy was written in the more resolved form

$$e = A\xi,$$

where A maps node variables to edge extensions. A constitutive matrix C then maps extensions to edge responses. The elastic energy is

$$V = \frac{1}{2}e^T C e = \frac{1}{2}\xi^T A^T C A \xi.$$

Therefore

$$K = A^T C A.$$

The hierarchy is

$$\xi \xrightarrow{A} e \xrightarrow{C} C e \xrightarrow{-A^T} \mathbf{F}_{\text{node}}.$$

This distinction is worth preserving even when the final equations of motion use only K .

- A is a node-to-edge compatibility/incidence relation.
- C carries constitutive information on the edges.
- A^T returns edge responses to nodal force balance.
- $K = A^T C A$ is a node-to-node operator.

Thus

$$(A, C) \implies K = A^T C A,$$

but, in general,

$$K \not\Rightarrow \text{a unique } A \text{ and } C.$$

Different edge-level realizations can produce the same effective node-to-node quadratic form.

If our only goal is to solve the nodal equations, K may contain enough information. If instead we want individual spring extensions, tensions, edge forces, or a local load path, then the factorization into A and C matters.

12.1 Electrical-network analogy

The same structure appears in a resistive electrical network. Let $\mathbf{v}$ be node voltages and let A be an oriented incidence matrix. Then

$$\mathbf{v}_{\text{branch}} = A\mathbf{v}$$

gives branch voltage differences. If G is a diagonal matrix of branch conductances,

$$\mathbf{i}_{\text{branch}} = G A \mathbf{v}.$$

Kirchhoff current balance at the nodes gives

$$A^T \mathbf{i}_{\text{branch}} = \mathbf{i}_{\text{injected}},$$

so the nodal equation is

$$\boxed{A^T G A \mathbf{v} = \mathbf{i}_{\text{injected}}.}$$

The nodal conductance matrix $A^T G A$ is analogous to the mechanical stiffness matrix K .

This is similar to reducing a circuit to a node-level description. The node equation can determine node voltages, but if we want branch currents we need the edge-level structure. The analogy should be used carefully: A^T directly expresses a KCL-type nodal balance, while $A\mathbf{v}$ constructs branch voltage differences from node potentials; their loop consistency is the corresponding KVL-type structure.

The broader lesson is that a second-order node operator can be a compressed description of a more resolved first-order edge structure.

13 The kinetic energy and the mass matrix

The general kinetic energy need not have the simple form

$$\frac{1}{2} m \dot{\boldsymbol{\xi}}^T \dot{\boldsymbol{\xi}}.$$

For generalized coordinates it can be written as

$$\boxed{T = \frac{1}{2} \dot{\mathbf{q}}^T M(\mathbf{q}) \dot{\mathbf{q}},}$$

where $M(\mathbf{q})$ is the configuration-dependent mass matrix or kinetic metric.

Near the equilibrium,

$$\mathbf{q} = \mathbf{q}^{(0)} + \boldsymbol{\xi}, \quad \dot{\mathbf{q}} = \dot{\boldsymbol{\xi}}.$$

Expand

$$M(\mathbf{q}) = M(\mathbf{q}^{(0)}) + O(\boldsymbol{\xi}).$$

Since $\dot{\boldsymbol{\xi}} = O(\epsilon)$ for a small oscillation, the correction

$$\dot{\boldsymbol{\xi}}^T [O(\boldsymbol{\xi})] \dot{\boldsymbol{\xi}}$$

is cubic in the small amplitude. Therefore the quadratic kinetic energy uses the mass matrix frozen at equilibrium:

$$\boxed{T^{(2)} = \frac{1}{2} \dot{\boldsymbol{\xi}}^T M \dot{\boldsymbol{\xi}}, \quad M \equiv M(\mathbf{q}^{(0)}).}$$

14 Why can M have off-diagonal entries?

In Cartesian particle coordinates, the mass matrix is usually diagonal. Off-diagonal terms appear naturally after changing to generalized coordinates because two generalized-coordinate motions may move some of the same physical masses.

Collect the Cartesian coordinates into a vector $\mathbf{x}$, with diagonal Cartesian mass matrix M_x :

$$T = \frac{1}{2} \dot{\mathbf{x}}^T M_x \dot{\mathbf{x}}.$$

Suppose

$$\mathbf{x} = \mathbf{x}(\mathbf{q}).$$

At a fixed configuration,

$$\dot{\mathbf{x}} = J_q \dot{\mathbf{q}}, \quad J_q \equiv \frac{\partial \mathbf{x}}{\partial \mathbf{q}^T}.$$

Therefore

$$T = \frac{1}{2} \dot{\mathbf{q}}^T J_q^T M_x J_q \dot{\mathbf{q}}.$$

Thus the generalized-coordinate mass matrix is

$$M_q = J_q^T M_x J_q.$$

Even if M_x is diagonal, M_q generally is not.

This also clarifies a feature of Lecture 10. There we deliberately chose the center-of-mass coordinate together with the relative coordinate so that the kinetic energy had no mixed term. A generic generalized-coordinate choice does not have that special property.

14.1 A two-coordinate example

Take two particles moving on a line and define

$$x_1 = q_1, \quad x_2 = q_1 + q_2.$$

Then

$$\dot{x}_1 = \dot{q}_1, \quad \dot{x}_2 = \dot{q}_1 + \dot{q}_2.$$

The kinetic energy is

$$\begin{aligned} T &= \frac{1}{2} m_1 \dot{q}_1^2 + \frac{1}{2} m_2 (\dot{q}_1 + \dot{q}_2)^2 \\ &= \frac{1}{2} \begin{pmatrix} \dot{q}_1 & \dot{q}_2 \end{pmatrix} \begin{pmatrix} m_1 + m_2 & m_2 \\ m_2 & m_2 \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix}. \end{aligned}$$

Hence

$$M = \begin{pmatrix} m_1 + m_2 & m_2 \\ m_2 & m_2 \end{pmatrix}.$$

The off-diagonal term has a simple physical meaning: changing either q_1 or q_2 contributes to the motion of particle 2.

15 The quadratic small-oscillation Lagrangian

Combining the quadratic kinetic and potential energies gives

$$L^{(2)} = \frac{1}{2} \dot{\boldsymbol{\xi}}^T M \dot{\boldsymbol{\xi}} - \frac{1}{2} \boldsymbol{\xi}^T K \boldsymbol{\xi}.$$

Here M and K are evaluated at the equilibrium and are constant matrices at this order.

The Euler–Lagrange equations give

$$M \ddot{\boldsymbol{\xi}} + K \boldsymbol{\xi} = \mathbf{0}.$$

This is the general linearized small-oscillation system.

Lecture 8 had the special case

$$M = mI,$$

for identical masses. The present equation shows what survives when the inertia is not proportional to the identity.

16 Normal modes and the generalized eigenvalue problem

Seek a motion with one fixed configuration-space pattern $\mathbf{u}$ and one scalar amplitude $a(t)$:

$$\boldsymbol{\xi}(t) = \mathbf{u} a(t).$$

Substituting into

$$M \ddot{\boldsymbol{\xi}} + K \boldsymbol{\xi} = \mathbf{0}$$

gives

$$M \mathbf{u} \ddot{a} + K \mathbf{u} a = \mathbf{0}.$$

If

$$\ddot{a} + \omega^2 a = 0,$$

then

$$K \mathbf{u} = \omega^2 M \mathbf{u}.$$

This is the generalized eigenvalue problem for small oscillations.

A useful physical reading is:

A normal-mode direction is a direction in configuration space for which the restoring-force pattern and the inertial response remain proportional, so the entire motion can share one scalar time dependence.

17 Why not simply diagonalize $M^{-1}K$?

If M is invertible, the generalized eigenvalue problem can certainly be written as

$$\boxed{M^{-1}K\mathbf{u} = \omega^2\mathbf{u}.}$$

So why retain the pair (K, M) ?

First, the two matrices have distinct physical meanings:

$$\boxed{M : \text{inertia}, \quad K : \text{restoring structure}.}$$

The generalized form keeps those roles visible.

Second, even when M and K are symmetric, $M^{-1}K$ is generally not symmetric in the ordinary Euclidean inner product. The natural modal geometry is mass weighted, not Euclidean.

Third, when M is positive definite, one can instead mass-weight the coordinates. Schematically, with

$$\mathbf{y} = M^{1/2}\mathbf{u},$$

the problem becomes

$$\boxed{M^{-1/2}KM^{-1/2}\mathbf{y} = \omega^2\mathbf{y}.}$$

The matrix on the left is symmetric when K is symmetric. This is the natural bridge to modal orthogonality and normal coordinates, which will be developed in Lecture 12.

If M happens to be diagonal, the rescaling is easy. But the generalized formulation remains conceptually useful because it makes clear which structure belongs to inertia and which belongs to the potential.

18 A familiar spring example with unequal masses

Consider two moving masses between fixed walls:

$$\text{wall} - \kappa - m_1 - \kappa - m_2 - \kappa - \text{wall}.$$

Let

$$\boldsymbol{\xi} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}.$$

Then

$$\boxed{M = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}}$$

and

$$\boxed{K = \kappa \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}}.$$

The normal modes satisfy

$$\boxed{\kappa \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \mathbf{u} = \omega^2 \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \mathbf{u}.}$$

The stiffness pattern is the same as in the equal-mass problem, but the two coordinates now carry different inertia.

If $m_1 = m_2 = m$, then $M = mI$ and the problem reduces to the ordinary stiffness eigenproblem of Lecture 8. If the masses differ, the mode shapes and frequencies depend on both K and M .

This example has diagonal M , which is convenient computationally, but the previous coordinate example shows that diagonal M is not guaranteed in general.

19 Zero modes require separate time dependence

The generalized eigenproblem may have a zero eigenvalue:

$$K\mathbf{u}_0 = 0.$$

Then

$$\omega_0 = 0,$$

and the scalar amplitude equation is

$$\ddot{a}_0 = 0.$$

Therefore

$$a_0(t) = a_0(0) + \dot{a}_0(0)t.$$

A zero-frequency mode is not necessarily static. It can represent uniform drift along a symmetry direction. The usual sine/cosine form for $\omega > 0$ must therefore be treated separately when $\omega = 0$.

20 What comes next

We have reached the general small-oscillation equation

$$M\ddot{\boldsymbol{\xi}} + K\boldsymbol{\xi} = \mathbf{0}$$

and the generalized eigenproblem

$$K\mathbf{u}_\alpha = \omega_\alpha^2 M\mathbf{u}_\alpha.$$

Lecture 12 will ask what this eigenproblem buys us structurally:

- What is the correct notion of orthogonality when $M \neq I$?
- Can we choose normal coordinates that separate both kinetic and potential energies?
- How do degeneracy and symmetry appear in the general problem?
- What do zero modes mean dynamically?
- How do higher-order terms limit the quadratic approximation?

The central comparison to carry forward is

central forces: exact symmetry reduction, conserved J, E , conic geometry,
 small oscillations: local quadratic approximation, matrices M, K , generalized modes.