Graduate Classical Mechanics
Lecture 3 Supplement
Geometric Products in Newtonian Mechanics

Power, turning, torque, angular momentum, and the virial identity

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1 Purpose of this supplement

The main Lecture 3 notes use ordinary vector calculus, with only brief references to geometric algebra. This supplement asks what additional structure becomes visible if, instead of taking only dot or cross products at selected moments, we temporarily keep the full Euclidean geometric product.

For vectors $\boldsymbol{a}$ and $\boldsymbol{b}$,

$$\boxed{\boldsymbol{ab} = \boldsymbol{a}\cdot\boldsymbol{b} + \boldsymbol{a}\wedge\boldsymbol{b}.}$$

The first term is a scalar. The second is a bivector, an oriented-plane quantity. We will not attempt a full development of geometric algebra here. The goal is narrower: to see how familiar Newtonian identities organize themselves when scalar and bivector information are kept together.

This is supplementary material. Nothing in the main course will require geometric-algebra notation unless it is introduced again explicitly.

2 The geometric product as symmetric plus antisymmetric information

Reversing the order of two vectors leaves the dot product unchanged and reverses the wedge product:

$$\boldsymbol{ba} = \boldsymbol{a}\cdot\boldsymbol{b} - \boldsymbol{a}\wedge\boldsymbol{b}.$$

Therefore

$$\boxed{\boldsymbol{a}\cdot\boldsymbol{b} = \frac{1}{2}(\boldsymbol{ab}+\boldsymbol{ba}), \qquad \boldsymbol{a}\wedge\boldsymbol{b} = \frac{1}{2}(\boldsymbol{ab}-\boldsymbol{ba}).}$$

The dot product is the symmetric part of the geometric product; the wedge product is the antisymmetric part.

For a single vector,

$$\boldsymbol{v}^2 = \boldsymbol{vv} = \boldsymbol{v}\cdot\boldsymbol{v} = v^2,$$

because $\boldsymbol{v}\wedge\boldsymbol{v} = 0$. The square of a Euclidean vector is therefore the ordinary scalar squared length.

3 Newton multiplied by velocity

Start with

$$\boldsymbol{F} = m\boldsymbol{a}, \qquad \boldsymbol{a} = \dot{\boldsymbol{v}}.$$

Ordinary work–energy theory keeps only the scalar part after multiplying by velocity:

$$\boldsymbol{v} \cdot \boldsymbol{F} = m\boldsymbol{v} \cdot \boldsymbol{a}.$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{v}^2) = \boldsymbol{a}\boldsymbol{v} + \boldsymbol{v}\boldsymbol{a} = 2\boldsymbol{a} \cdot \boldsymbol{v},$$

we recover

$$\boxed{\boldsymbol{v} \cdot \boldsymbol{F} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}mv^2\right) = \dot{T}.}$$

In this language, the kinetic-energy identity is the *symmetric* part of multiplying Newton's equation by velocity.

But the full product is

$$\boxed{\boldsymbol{v}\boldsymbol{F} = \boldsymbol{v} \cdot \boldsymbol{F} + \boldsymbol{v} \wedge \boldsymbol{F}.}$$

The scalar part is power. What does the bivector part mean?

3.1 The bivector part measures turning

Write the velocity as

$$\boldsymbol{v} = v\boldsymbol{t},$$

where $\boldsymbol{t}$ is the unit tangent to the trajectory. In planar motion let $\boldsymbol{n}$ be the oriented unit normal and let ψ be the heading angle. Then

$$\dot{\boldsymbol{t}} = \dot{\psi}\,\boldsymbol{n},$$

and

$$\boldsymbol{a} = \dot{v}\,\boldsymbol{t} + v\dot{\psi}\,\boldsymbol{n}.$$

Therefore

$$\boldsymbol{v} \wedge \boldsymbol{F} = m\boldsymbol{v} \wedge \boldsymbol{a} = mv^2\dot{\psi}\,\boldsymbol{t} \wedge \boldsymbol{n}.$$

Hence

$$\boxed{\boldsymbol{v} \wedge \boldsymbol{F} = mv^2\dot{\psi}\,\boldsymbol{t} \wedge \boldsymbol{n}.}$$

Its magnitude is controlled by the transverse force; its orientation records the oriented plane in which the velocity turns.

The two grades of $\boldsymbol{v}\boldsymbol{F}$ therefore separate two physically distinct effects:

$$\boxed{\begin{array}{rcl} \boldsymbol{v} \cdot \boldsymbol{F} & : & \text{change of kinetic energy,} \\ \boldsymbol{v} \wedge \boldsymbol{F} & : & \text{turning of the velocity direction.} \end{array}}$$

An ideal normal constraint is a particularly clean example. If $\boldsymbol{F}_N \perp \boldsymbol{v}$, then

$$\boldsymbol{v} \cdot \boldsymbol{F}_N = 0$$

while generally

$$\boldsymbol{v} \wedge \boldsymbol{F}_N \neq 0.$$

The constraint can bend the trajectory without doing work.

Order matters. The scalar part does not care whether we write $\boldsymbol{v}\boldsymbol{F}$ or $\boldsymbol{F}\boldsymbol{v}$, but

$$\boldsymbol{F} \wedge \boldsymbol{v} = -\boldsymbol{v} \wedge \boldsymbol{F}.$$

Thus reversing the order reverses the orientation of the bivector part. This is not a nuisance; it is the algebra remembering orientation.

4 Conservative forces and exact work

The geometric product does not replace the exactness criterion for conservative force. Potential energy is tied specifically to the scalar work form

$$\omega = \boldsymbol{F}(\boldsymbol{x}) \cdot \mathrm{d}\boldsymbol{x}.$$

A force is conservative on a region when this one-form is exact:

$$\boxed{\omega = -\mathrm{d}V.}$$

Then

$$\boldsymbol{F} = -\nabla V$$

and

$$\int_A^B \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{x} = -[V(B) - V(A)].$$

Consequently

$$\oint \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{x} = 0.$$

In differential-form language, exactness implies closedness:

$$\omega = -\mathrm{d}V \quad \Longrightarrow \quad \mathrm{d}\omega = 0,$$

because $\mathrm{d}^2 = 0$. Generalized Stokes then gives

$$\oint_{\partial\Sigma} \omega = \int_{\Sigma} \mathrm{d}\omega = 0.$$

In ordinary three-dimensional vector calculus, $\mathrm{d}\omega = 0$ corresponds to

$$\nabla \times \boldsymbol{F} = \boldsymbol{0}.$$

On a simply connected region, this local condition is sufficient for a global potential. On a region with nontrivial topology, closed need not imply exact; the distinction will matter when topology enters more seriously.

This gives a useful hierarchy:

$$\boxed{\text{exact work} \Longrightarrow \text{endpoint dependence} \Longrightarrow \text{zero closed-loop work.}}$$

The converse statements require appropriate assumptions on the domain.

5 Newton multiplied by position

Now multiply the force by position instead of velocity:

$$\boxed{\boldsymbol{x}\boldsymbol{F} = \boldsymbol{x} \cdot \boldsymbol{F} + \boldsymbol{x} \wedge \boldsymbol{F}.}$$

Again the scalar and bivector parts have familiar but different physical meanings.

5.1 Bivector part: torque and angular momentum

Define the angular-momentum bivector

$$\boxed{\boldsymbol{L}_{\text{biv}} = \boldsymbol{x} \wedge \boldsymbol{p}, \qquad \boldsymbol{p} = m\boldsymbol{v}.}$$

Then

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{x} \wedge \boldsymbol{p}) &= \boldsymbol{v} \wedge \boldsymbol{p} + \boldsymbol{x} \wedge \dot{\boldsymbol{p}} \\ &= m\boldsymbol{v} \wedge \boldsymbol{v} + \boldsymbol{x} \wedge \boldsymbol{F} \\ &= \boldsymbol{x} \wedge \boldsymbol{F}. \end{aligned}$$

Thus

$$\boxed{\dot{\boldsymbol{L}}_{\text{biv}} = \boldsymbol{\tau}_{\text{biv}}, \qquad \boldsymbol{\tau}_{\text{biv}} = \boldsymbol{x} \wedge \boldsymbol{F}.}$$

The usual three-dimensional axial-vector equations

$$\vec{L} = \boldsymbol{x} \times \boldsymbol{p}, \qquad \vec{\tau} = \boldsymbol{x} \times \boldsymbol{F}$$

are a three-dimensional representation of these oriented-plane objects.

For a central force, $\boldsymbol{F} \parallel \boldsymbol{x}$, so

$$\boldsymbol{x} \wedge \boldsymbol{F} = 0$$

and angular momentum is conserved immediately.

5.2 Scalar part: the virial quantity

Define

$$G = \boldsymbol{x} \cdot \boldsymbol{p}.$$

Then

$$\begin{aligned} \dot{G} &= \boldsymbol{v} \cdot \boldsymbol{p} + \boldsymbol{x} \cdot \dot{\boldsymbol{p}} \\ &= mv^2 + \boldsymbol{x} \cdot \boldsymbol{F} \\ &= 2T + \boldsymbol{x} \cdot \boldsymbol{F}. \end{aligned}$$

Therefore

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{x} \cdot \boldsymbol{p}) = 2T + \boldsymbol{x} \cdot \boldsymbol{F}.}$$

The scalar part of $\boldsymbol{x}\boldsymbol{F}$ is thus the force term in the virial identity.

For bounded motion, a long-time average often makes the average of $\dot{G}$ vanish, giving

$$2\langle T\rangle = -\langle \boldsymbol{x} \cdot \boldsymbol{F}\rangle.$$

If the potential is homogeneous of degree k,

$$V(\lambda \boldsymbol{x}) = \lambda^k V(\boldsymbol{x}),$$

Euler's theorem gives

$$\boldsymbol{x} \cdot \nabla V = kV.$$

For $\boldsymbol{F} = -\nabla V$,

$$\boldsymbol{x} \cdot \boldsymbol{F} = -kV,$$

and therefore

$$\boxed{2\langle T\rangle = k\langle V\rangle.}$$

This is the familiar virial theorem, emerging from the scalar part of the position–force product.

6 One equation containing both virial and torque

The previous two identities can be packaged together. Since

$$\boldsymbol{p} = m\boldsymbol{v},$$

we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{x}\boldsymbol{p}) &= \boldsymbol{v}\boldsymbol{p} + \boldsymbol{x}\dot{\boldsymbol{p}} \\
&= m\boldsymbol{v}\boldsymbol{v} + \boldsymbol{x}\boldsymbol{F} \\
&= 2T + \boldsymbol{x}\boldsymbol{F}.
\end{aligned}$$

Hence

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{x}\boldsymbol{p}) = 2T + \boldsymbol{x}\boldsymbol{F}.}$$

Taking the scalar part gives

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{x} \cdot \boldsymbol{p}) = 2T + \boldsymbol{x} \cdot \boldsymbol{F},$$

while taking the bivector part gives

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{x} \wedge \boldsymbol{p}) = \boldsymbol{x} \wedge \boldsymbol{F}.$$

Thus one geometric-product equation contains both identities.

This is perhaps the most compelling reason to keep the geometric product in view: familiar scalar and rotational balance laws appear as different grades of the same algebraic statement.

7 A compact comparison

Product	Scalar part	Bivector part
$\boldsymbol{vF}$	power, $\boldsymbol{v} \cdot \boldsymbol{F} = \dot{T}$	turning, $\boldsymbol{v} \wedge \boldsymbol{F}$
$\boldsymbol{xF}$	virial term, $\boldsymbol{x} \cdot \boldsymbol{F}$	torque, $\boldsymbol{x} \wedge \boldsymbol{F}$
$\boldsymbol{xp}$	virial quantity, $\boldsymbol{x} \cdot \boldsymbol{p}$	angular momentum, $\boldsymbol{x} \wedge \boldsymbol{p}$

The pattern is suggestive:

$$\boxed{\begin{array}{c}\text{scalar grade} \longleftrightarrow \text{energetic/longitudinal information,} \\ \text{bivector grade} \longleftrightarrow \text{oriented-plane/rotational information.}\end{array}}$$

This is a useful organizing principle, but not a theorem that every scalar quantity is "energy" or every bivector quantity is "rotation." The physical interpretation still comes from the dynamical context.

8 Where this viewpoint may reappear in the course

The geometric-product language can be introduced lightly now and developed only when it clarifies structure:

- **Constraints:** $\boldsymbol{v} \cdot \boldsymbol{F}_N = 0$ but $\boldsymbol{v} \wedge \boldsymbol{F}_N \neq 0$ cleanly separates no-work constraints from trajectory turning.
- **Central forces:** $\boldsymbol{x} \wedge \boldsymbol{F} = 0$ gives angular-momentum conservation without first replacing the oriented plane by an axial vector.
- **Rigid-body motion:** bivectors and rotors offer a coordinate-free alternative to treating rotations primarily through cross products and pseudovectors.
- **Generalized Stokes:** exactness of the scalar work one-form belongs naturally with exterior differentiation and the principle $\partial^2 = 0$.
- **Virial arguments:** the scalar part of $\boldsymbol{xF}$ provides a natural bridge to scaling arguments and homogeneous potentials.

A reasonable course-design strategy is therefore not to turn the course into a geometric-algebra course, but to use geometric products selectively when they reveal that two familiar equations are grades of one underlying statement.

9 Summary

Three structural observations are worth retaining:

1. Multiplying Newton's equation by velocity and taking the symmetric/scalar part produces the work–energy identity.

2. The remaining bivector part of $\boldsymbol{vF}$ records transverse turning information that the power alone discards.

3. The single identity

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{xp}) = 2T + \boldsymbol{xF}$$

   contains both the virial equation and the torque–angular-momentum equation as its scalar and bivector parts.

Potential energy enters through a complementary idea: the scalar work form $\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{x}$ is conservative precisely when it is exact. Together, exactness and geometric grading provide two different ways of making hidden Newtonian structure visible.