Graduate Classical Mechanics

Lecture 3: Newtonian Mechanics, Vectors, and the Structure of Force

A review that prepares the transition to Lagrangian mechanics

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1 Why pause for Newtonian mechanics?

The first two lectures emphasized states, transitions, discrete differences, boundaries, and the matrix structure

$$A^{\mathsf{T}} C A.$$

Before introducing the action principle, it is useful to make sure that we have a common language for ordinary Newtonian mechanics and for the mathematical objects that appear in it.

This is not intended as a complete review of undergraduate mechanics. The purpose is to isolate the pieces that will matter when we move to Lagrangian and Hamiltonian formulations:

- what a vector is, independent of a coordinate representation;
- what additional structure is supplied by an inner product;
- what we mean by the Newtonian state of a system;
- force, work, exactness, potential energy, and energy conservation;
- many-particle systems and conservation laws;
- equilibrium and linearization;
- constraints and why generalized coordinates are useful.

Throughout the course we will deliberately use different notation for the three major formulations:

Formulation	Configuration variables	Natural state variables
Newtonian	$\boldsymbol{x}, \dot{\boldsymbol{x}}$	$(\boldsymbol{x}, \dot{\boldsymbol{x}})$
Lagrangian	$\boldsymbol{q}, \dot{\boldsymbol{q}}$	$(\boldsymbol{q}, \dot{\boldsymbol{q}})$
Hamiltonian	$\boldsymbol{q}, \boldsymbol{p}$	$(\boldsymbol{q}, \boldsymbol{p})$

The generalized coordinate q is a generalization of the Cartesian coordinate x; in a simple problem we may have $q = x$. The notation is meant to remind us which formulation is doing the conceptual work.

2 What is a vector?

There are two common ways to introduce vectors. We might say that a vector is "something with magnitude and direction," or we might represent a vector as a list of numbers. Neither description captures what makes something a vector.

A more useful starting point is:

> **A vector is something you can add and scale like real numbers.**

This is not yet a formal definition, but it is a useful mnemonic for the formal one. The phrase "like real numbers" means that addition and scaling obey the familiar algebraic rules. In particular, vectors can be added to one another, vectors can be multiplied by real scalars, and these operations satisfy the corresponding associativity, commutativity, identity, inverse, and distributive laws.

For example,

$$\boldsymbol{v} + \mathbf{0} = \boldsymbol{v}, \qquad \boldsymbol{v} + (-\boldsymbol{v}) = \mathbf{0},$$

and

$$1\boldsymbol{v} = \boldsymbol{v}, \qquad a(b\boldsymbol{v}) = (ab)\boldsymbol{v},$$

with

$$a(\boldsymbol{u} + \boldsymbol{v}) = a\boldsymbol{u} + a\boldsymbol{v}.$$

The formal definition of a vector space makes these rules precise. What matters here is that the definition does not require vectors to be arrows or lists of numbers. Polynomials, matrices, functions, and many other objects can be vectors whenever they support the same addition-and-scaling structure.

Thus, as a first mental model,

$$\boxed{\text{vector} = \text{something you can add and scale like real numbers.}}$$

2.1 A vector is not its list of components

Choose a basis $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$. Then a vector can be represented as

$$\boldsymbol{v} = v^1\boldsymbol{e}_1 + \cdots + v^n\boldsymbol{e}_n.$$

The column

$$[\boldsymbol{v}]_{\mathcal{B}} = \begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix}$$

is a representation of the vector relative to the basis $\mathcal{B}$.

If the basis changes, the numbers change even though the vector does not. This is the same distinction we will later make between a physical state and a coordinate description of that state.

> **The vector is the object; the component list is a representation of that object in a chosen basis.**

2.2 Magnitude and direction require additional structure

The phrase "magnitude and direction" already assumes more than the vector-space axioms. To define a magnitude we need an inner product. In Euclidean space,

$$\|\boldsymbol{v}\|^2 = \boldsymbol{v} \cdot \boldsymbol{v}.$$

The same inner product allows us to define angles by

$$\boldsymbol{u} \cdot \boldsymbol{v} = \|\boldsymbol{u}\| \, \|\boldsymbol{v}\| \cos\theta.$$

So the logical order is

$$\boxed{\text{vector space} \longrightarrow \text{basis if desired} \longrightarrow \text{inner product if geometry is desired.}}$$

This matters in mechanics because we frequently change coordinates while keeping the underlying geometric object fixed.

2.3 Points and displacement vectors

It is also useful to distinguish a point from a vector. If P and Q are points, then

$$Q - P$$

is naturally a displacement vector. Once an origin O is selected, a point P can be represented by the position vector

$$\boldsymbol{x} = P - O.$$

The choice of origin is part of the representation, not part of the physical point itself. The mathematical language for this distinction is an *affine space*, although we will not need to develop that formalism in detail.

3 A brief geometric-algebra viewpoint

The vector-space definition gives addition and scalar multiplication. It does not provide a multiplication of vectors.

The dot product supplies one additional operation,

$$\boldsymbol{u} \cdot \boldsymbol{v},$$

but its result is a scalar and it loses information about the oriented plane spanned by the two vectors.

The familiar three-dimensional cross product,

$$\boldsymbol{u} \times \boldsymbol{v},$$

packages that oriented plane as an axial vector perpendicular to the plane. This is very useful in three-dimensional mechanics, but it is special to three dimensions.

A geometric-algebra viewpoint keeps the oriented plane itself. One writes the geometric product schematically as

$$\boxed{\boldsymbol{u}\boldsymbol{v} = \boldsymbol{u}\cdot\boldsymbol{v} + \boldsymbol{u}\wedge\boldsymbol{v},}$$

where

$$\boldsymbol{u}\wedge\boldsymbol{v}$$

is a bivector: an oriented-plane quantity.

We will not develop geometric algebra as a separate formalism in this course. We will continue to use ordinary vector notation and the cross product. But when the geometry is especially useful, we will occasionally point out the object that the conventional notation is representing.

For example, torque can be viewed fundamentally as

$$\boldsymbol{\tau} = \boldsymbol{x}\wedge\boldsymbol{F},$$

an oriented plane, while the usual three-dimensional representation is

$$\vec{\tau} = \boldsymbol{x}\times\boldsymbol{F}.$$

This interpretation will return when we discuss angular momentum and rigid-body rotations.

4 The Newtonian description

Consider N particles in ordinary Euclidean space. Let

$$\boldsymbol{x}_n(t)$$

denote the Cartesian position of particle n and

$$\dot{\boldsymbol{x}}_n(t)$$

its velocity.

The Newtonian input is a force law

$$\boldsymbol{F}_n = \boldsymbol{F}_n(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N,\dot{\boldsymbol{x}}_1,\ldots,\dot{\boldsymbol{x}}_N,t),$$

and Newton's second law is

$$\boxed{m_n\ddot{\boldsymbol{x}}_n = \boldsymbol{F}_n.}$$

For a single particle this is often written

$$m\ddot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x},\dot{\boldsymbol{x}},t).$$

4.1 Position alone is not the state

Lecture 1 emphasized that a useful state must contain enough information for the dynamics to close. The equation

$$m\ddot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x}, \dot{\boldsymbol{x}}, t)$$

is second order in position. Knowing only $\boldsymbol{x}(t_0)$ does not determine the future.

A Newtonian state can instead be taken to be

$$\boxed{S = (\boldsymbol{x}, \dot{\boldsymbol{x}}).}$$

Then the dynamics are first order on the state space:

$$\frac{\mathrm{d}S}{\mathrm{d}t} = f(S,t) = \begin{pmatrix} \dot{\boldsymbol{x}} \\ m^{-1}\boldsymbol{F}(\boldsymbol{x}, \dot{\boldsymbol{x}}, t) \end{pmatrix}.$$

This lets us make the connection to Lecture 1 completely explicit. There we began with the finite evolution rule

$$\boxed{S_{n+1} = F(S_n).}$$

The differential and finite-update descriptions can be related in two logically distinct ways.

Viewpoint 1: finite updates are fundamental

Suppose the underlying evolution is a sequence of states

$$S_n \longrightarrow S_{n+1}, \qquad \Delta t_n = t_{n+1} - t_n,$$

governed by an exact update rule

$$S_{n+1} = F(S_n).$$

If successive states differ only slightly, then a continuum description may approximate the finite change by

$$\left.\frac{\mathrm{d}S}{\mathrm{d}t}\right|_{t_n} \approx \frac{S_{n+1} - S_n}{\Delta t_n} = \frac{F(S_n) - S_n}{\Delta t_n}.$$

Writing the continuum approximation as

$$\frac{\mathrm{d}S}{\mathrm{d}t} = f(S,t),$$

we recover, to first order in the step,

$$\boxed{S_{n+1} \approx S_n + \Delta t_n\, f(S_n, t_n).}$$

Thus the derivative may be viewed as a local, compressed description of an underlying finite change.

Viewpoint 2: continuous evolution is fundamental

Instead suppose the differential equation

$$\frac{\mathrm{d}S}{\mathrm{d}t} = f(S,t)$$

is the fundamental law. Integrating from t_n to t_{n+1} gives the exact integral equation

$$S(t_{n+1}) = S(t_n) + \int_{t_n}^{t_{n+1}} f(S(t),t)\,\mathrm{d}t.$$

For a well-posed initial-value problem, this evolution defines a finite-time flow map. If the system is autonomous, so that $f = f(S)$, we may write

$$\boxed{S_{n+1} = \Phi_{\Delta t}(S_n),} \qquad \Delta t = t_{n+1} - t_n.$$

Calling the flow map F gives exactly the Lecture 1 form

$$\boxed{S_{n+1} = F(S_n).}$$

In this viewpoint the finite map need not be an approximation: it is the exact finite-time evolution generated by the differential equation. What is approximate is replacing that exact flow by, for example,

$$\Phi_{\Delta t}(S_n) \approx S_n + \Delta t\, f(S_n),$$

which is the forward-Euler approximation.

The two viewpoints therefore meet at the same structural statement:

$$\boxed{\text{a complete state contains enough information to determine the next state.}}$$

The notation $S_{n+1} = F(S_n)$ does not by itself decide whether nature is fundamentally discrete or continuous; it expresses closure of the dynamics on the chosen state.

There is one bookkeeping issue when the force depends explicitly on time. If

$$\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{x}, \dot{\boldsymbol{x}}, t),$$

then the map on $(\boldsymbol{x}, \dot{\boldsymbol{x}})$ is generally time dependent,

$$S_{n+1} = F_n(S_n).$$

Alternatively we may enlarge the state to include the clock variable,

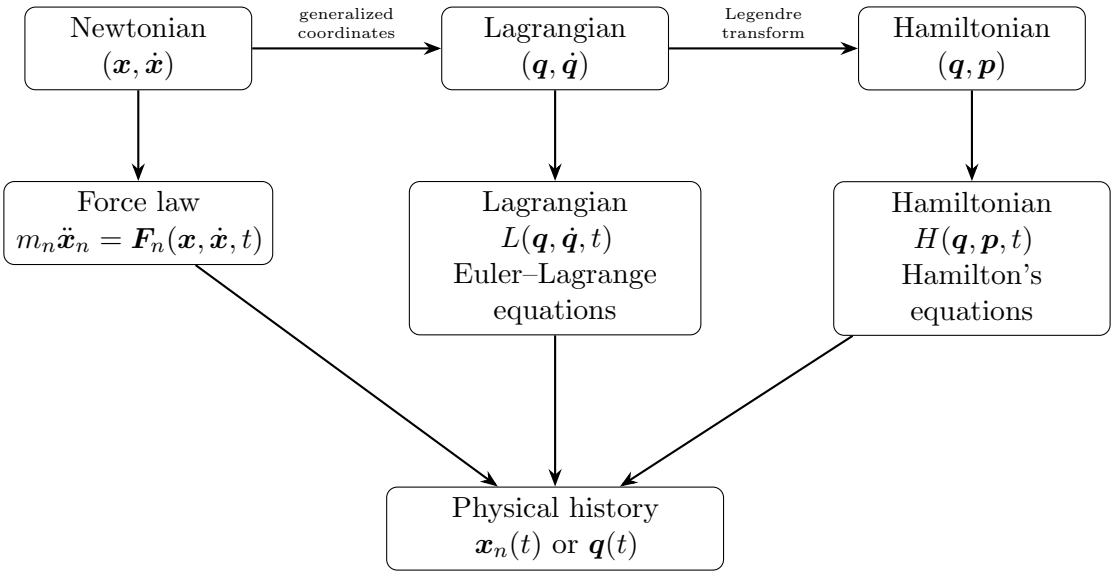
$$\widetilde{S} = (\boldsymbol{x}, \dot{\boldsymbol{x}}, t), \qquad \frac{\mathrm{d}\widetilde{S}}{\mathrm{d}t} = \begin{pmatrix} \dot{\boldsymbol{x}} \\ m^{-1}\boldsymbol{F}(\boldsymbol{x}, \dot{\boldsymbol{x}}, t) \\ 1 \end{pmatrix},$$

and recover an autonomous map

$$\widetilde{S}_{n+1} = \widetilde{F}(\widetilde{S}_n).$$

5 An overview of the three formulations

The following diagram is adapted from the review structure used in the earlier undergraduate mechanics course. It is not meant to define the theories completely; rather, it shows how the three formulations take different inputs while ultimately describing the same physical history.



The point of the course is not merely to learn three ways of solving the same differential equation. Each formulation makes different structure visible.

6 Internal and external forces

For a system of particles it is often useful to separate the force on particle n into forces from the other particles and forces from outside the system:

$$\boxed{m_n\ddot{\boldsymbol{x}}_n = \sum_{k\neq n} \boldsymbol{F}_{nk}^{\text{int}} + \boldsymbol{F}_n^{\text{ext}}.}$$

The weak form of Newton's third law states that pair forces are equal and opposite,

$$\boldsymbol{F}_{nk}^{\text{int}} = -\boldsymbol{F}_{kn}^{\text{int}}.$$

If we sum Newton's equation over all particles, the internal pair forces cancel. Defining

$$M = \sum_n m_n, \qquad \boldsymbol{X}_{\text{cm}} = \frac{1}{M}\sum_n m_n\boldsymbol{x}_n,$$

we obtain

$$\boxed{M\ddot{\boldsymbol{X}}_{\text{cm}} = \sum_n \boldsymbol{F}_n^{\text{ext}}.}$$

This is the many-particle analogue of the cancellation of internal spring forces discussed in Lecture 2. In both cases, internal interactions disappear from the balance for the whole system.

6.1 Angular momentum and the strong third law

If internal forces are also central, so that the force between two particles lies along their separation, the pair has zero net internal torque. Then the total angular momentum about a fixed origin,

$$\boldsymbol{L} = \sum_n (\boldsymbol{x}_n - \boldsymbol{x}_0) \times m_n \dot{\boldsymbol{x}}_n,$$

satisfies

$$\boxed{\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} = \boldsymbol{\tau}_{\mathrm{ext}}.}$$

Later we will reinterpret angular momentum and torque geometrically, and we will derive conservation laws from symmetry rather than from pairwise force arguments.

7 Work, kinetic energy, and exactness

For one particle,

$$m\ddot{\boldsymbol{x}} = \boldsymbol{F}.$$

Contract both sides with the velocity:

$$\boldsymbol{F} \cdot \dot{\boldsymbol{x}} = m\ddot{\boldsymbol{x}} \cdot \dot{\boldsymbol{x}}.$$

The right-hand side is not merely an expression we know how to integrate; it is an *exact time derivative*:

$$m\ddot{\boldsymbol{x}} \cdot \dot{\boldsymbol{x}} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2} m \dot{\boldsymbol{x}}^{\,2}\right).$$

This observation identifies the kinetic energy

$$\boxed{T = \frac{1}{2} m \dot{\boldsymbol{x}}^{\,2}, \qquad \frac{\mathrm{d}T}{\mathrm{d}t} = \boldsymbol{F} \cdot \dot{\boldsymbol{x}}.}$$

Thus the scalar product of force with velocity is the instantaneous power,

$$P = \boldsymbol{F} \cdot \dot{\boldsymbol{x}}.$$

Along the actual trajectory, $\mathrm{d}\boldsymbol{x} = \dot{\boldsymbol{x}}\,\mathrm{d}t$, so

$$\boxed{\mathrm{d}T = \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{x}.}$$

Integrating gives the work–energy relation

$$T_B - T_A = \int_A^B \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{x}.$$

The important point is the logic: kinetic energy appears because the Newtonian expression $m\ddot{\boldsymbol{x}} \cdot \dot{\boldsymbol{x}}$ is recognized as an exact derivative. We can now ask the symmetric question on the force side.

8 Conservative forces: exactness on configuration space

Suppose first that the relevant force is a position-dependent vector field $\boldsymbol{F}(\boldsymbol{x})$. Its infinitesimal work is the work one-form

$$\omega = \boldsymbol{F}(\boldsymbol{x}) \cdot \mathrm{d}\boldsymbol{x}.$$

Rather than beginning by postulating a potential, ask:

> **When is the work form itself an exact differential of a scalar function of position?**

If there exists a scalar $V(\boldsymbol{x})$ such that

$$\boxed{\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{x} = -\mathrm{d}V,}$$

then the force is conservative on that region. Since

$$\mathrm{d}V = \nabla V \cdot \mathrm{d}\boldsymbol{x},$$

exactness implies

$$\boxed{\boldsymbol{F} = -\nabla V.}$$

Thus $\boldsymbol{F} = -\nabla V$ is not an unrelated extra assumption: it is the coordinate expression of the statement that the work form is exact.

Now the symmetry with the kinetic-energy side is visible:

$$\underbrace{m\ddot{\boldsymbol{x}} \cdot \dot{\boldsymbol{x}}}_{\mathrm{d}T/\mathrm{d}t} = \underbrace{\boldsymbol{F} \cdot \dot{\boldsymbol{x}}}_{-\mathrm{d}V/\mathrm{d}t}.$$

Therefore

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}(T+V) = 0, \qquad E = T + V = \text{constant}.}$$

For a conservative, time-independent system, both sides of Newton's equation have been converted into exact derivatives.

Exactness also explains path independence. If

$$\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{x} = -\mathrm{d}V,$$

then

$$\int_A^B \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{x} = -\big[V(B) - V(A)\big],$$

so the work depends only on the endpoints, and

$$\oint \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{x} = 0.$$

This is another appearance of the boundary principle from Lecture 2: accumulated local change collapses to boundary data. In a simply connected region of ordinary three-dimensional space, the same condition can be expressed locally as

$$\nabla \times \boldsymbol{F} = \boldsymbol{0}.$$

We will later recognize this as part of the broader Stokes structure.

Explicit time dependence and nonconservative forces. If $V = V(\boldsymbol{x}, t)$ and $\boldsymbol{F}_{\text{cons}} = -\nabla V$, then

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \nabla V \cdot \dot{\boldsymbol{x}} + \frac{\partial V}{\partial t},$$

and hence

$$\frac{\mathrm{d}}{\mathrm{d}t}(T + V) = \frac{\partial V}{\partial t}$$

when no other forces are present. If an additional nonconservative force $\boldsymbol{F}_{\text{nc}}$ acts, then

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}(T + V) = \frac{\partial V}{\partial t} + \boldsymbol{F}_{\text{nc}} \cdot \dot{\boldsymbol{x}}.}$$

This form makes the bookkeeping transparent: explicit time dependence and nonconservative power are the channels through which the reduced mechanical energy changes.

Geometric-product aside: what did the dot product leave out?

Earlier we briefly introduced the Euclidean geometric product

$$\boldsymbol{ab} = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \wedge \boldsymbol{b}.$$

If, instead of keeping only the scalar contraction with the velocity, we look at the full product, then

$$\boxed{\dot{\boldsymbol{x}}\,\boldsymbol{F} = \dot{\boldsymbol{x}} \cdot \boldsymbol{F} + \dot{\boldsymbol{x}} \wedge \boldsymbol{F}.}$$

The scalar part is the power. The bivector part measures the oriented transverse action of the force and therefore encodes the tendency of the velocity to turn. An ideal normal constraint is a useful example: it can satisfy

$$\boldsymbol{F} \cdot \dot{\boldsymbol{x}} = 0$$

while still having

$$\dot{\boldsymbol{x}} \wedge \boldsymbol{F} \neq 0.$$

It changes the direction of motion without changing the kinetic energy.

Likewise,

$$\boxed{\boldsymbol{x}\boldsymbol{F} = \boldsymbol{x} \cdot \boldsymbol{F} + \boldsymbol{x} \wedge \boldsymbol{F}.}$$

The bivector part $\boldsymbol{x} \wedge \boldsymbol{F}$ is torque, while the scalar part $\boldsymbol{x} \cdot \boldsymbol{F}$ is the quantity that enters the virial identity. We will continue to use ordinary vector notation in the main course, but the separate Lecture 3 supplement develops these geometric-product identities in more detail.

This is our first example of a scalar function encoding a vector force field. That compression will become even more important in Lagrangian mechanics, where a single scalar function L generates all of the equations of motion.

9 Equilibrium and linearization

Suppose $\boldsymbol{x}^{(0)}$ is an equilibrium configuration for a conservative system:

$$\nabla V\big|_{\boldsymbol{x}^{(0)}} = 0.$$

Define displacement from equilibrium by

$$\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{x}^{(0)}.$$

Taylor expand the potential:

$$V(\boldsymbol{x}^{(0)} + \boldsymbol{u}) = V(\boldsymbol{x}^{(0)}) + \underbrace{\nabla V|_{\boldsymbol{x}^{(0)}} \cdot \boldsymbol{u}}_{0} + \frac{1}{2}\boldsymbol{u}^{\mathsf{T}} H \boldsymbol{u} + \mathcal{O}(\|\boldsymbol{u}\|^3),$$

where the Hessian is

$$H_{ij} = \left.\frac{\partial^2 V}{\partial x_i \partial x_j}\right|_{\boldsymbol{x}^{(0)}}.$$

Thus near equilibrium,

$$\boxed{V \simeq V_0 + \frac{1}{2}\boldsymbol{u}^{\mathsf{T}} H \boldsymbol{u},}$$

and

$$\boxed{\boldsymbol{F}_{\text{int}} = -\nabla V \simeq -H\boldsymbol{u}.}$$

This connects directly to Lecture 2. For the linear spring network,

$$V(\boldsymbol{u}) = \frac{1}{2}\boldsymbol{u}^{\mathsf{T}} A^{\mathsf{T}} C A \boldsymbol{u},$$

so

$$\boxed{H = A^{\mathsf{T}} C A.}$$

The matrix we built from incidence and constitutive information is therefore the Hessian of the potential energy at equilibrium.

For a mass matrix M, Newton's equations near equilibrium become

$$\boxed{M\ddot{\boldsymbol{u}} + H\boldsymbol{u} = \boldsymbol{f},}$$

or, for the spring system,

$$\boxed{M\ddot{\boldsymbol{u}} + A^{\mathsf{T}} C A \boldsymbol{u} = \boldsymbol{f}.}$$

We will postpone the eigenvalue problem associated with this equation until our discussion of small oscillations.

10 Constraints and the limits of Cartesian Newtonian mechanics

Newton's laws are completely adequate in principle, but constrained systems can make a force-based formulation cumbersome.

Consider a particle constrained to move on a surface

$$f(\boldsymbol{x}) = 0.$$

Newton's equation becomes

$$m\ddot{\boldsymbol{x}} = \boldsymbol{F}_{\text{applied}} + \boldsymbol{F}_{\text{constraint}}.$$

The constraint force must be chosen so that the trajectory remains on the allowed surface. Often this force is not physically interesting; it exists only to enforce the constraint.

For several particles the number of Cartesian coordinates may greatly exceed the number of independent degrees of freedom. A double pendulum in a plane, for example, can be described by four Cartesian coordinates,

$$(x_1, y_1, x_2, y_2),$$

but two rigid-length constraints reduce the system to two independent coordinates. A natural choice is

$$(q_1, q_2) = (\theta_1, \theta_2).$$

This motivates the Lagrangian language:

$$\boxed{\boldsymbol{x}_n = \boldsymbol{x}_n(\boldsymbol{q}, t),}$$

where the generalized coordinates $\boldsymbol{q}$ are chosen to describe the allowed configuration directly.

For holonomic constraints, the dimension of $\boldsymbol{q}$ is the number of independent degrees of freedom after the constraints have been imposed. We do not need to carry the eliminated constraint forces as independent unknowns.

11 Why move to Lagrangian mechanics?

Newtonian mechanics begins with a vector force law,

$$m\ddot{\boldsymbol{x}} = \boldsymbol{F}.$$

This formulation is powerful and intuitive, but several difficulties become increasingly visible:

- Cartesian coordinates may not match the natural degrees of freedom;
- constraint forces may introduce variables that we do not actually care about;
- conservation laws can appear as separate consequences of different force arguments;
- the geometric structure of the configuration space is not always obvious.

The Lagrangian formulation replaces the direct force description by a scalar function

$$L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t),$$

and ultimately by the action associated with an entire path.

Before taking a continuum variation, however, we will follow the philosophy developed in the first two lectures and begin with a finite path:

$$\boldsymbol{q}_0, \boldsymbol{q}_1, \ldots, \boldsymbol{q}_N.$$

The next lecture asks:

> **Can the dynamics of a path be obtained from a scalar quantity associated with the whole path, rather than from a vector force balance at every instant?**

That question leads to the discrete action and the variational principle.

Connections to keep in mind

1. A vector is not its coordinates; the coordinates depend on a chosen basis.
2. "Magnitude and direction" requires an inner product and therefore more structure than a vector space alone.
3. Newtonian mechanics will use $\boldsymbol{x}$ and $\dot{\boldsymbol{x}}$; Lagrangian mechanics will use generalized $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$; Hamiltonian mechanics will use $\boldsymbol{q}$ and $\boldsymbol{p}$.
4. A second-order equation for $\boldsymbol{x}$ is a first-order evolution equation for the complete Newtonian state $(\boldsymbol{x}, \dot{\boldsymbol{x}})$.
5. Conservative forces compress into a scalar potential: $\boldsymbol{F} = -\nabla V$.
6. Near equilibrium, the Hessian of V is the stiffness matrix. For our spring network, $H = A^{\mathsf{T}} C A$.
7. Constraints motivate coordinates adapted to the actual degrees of freedom.
8. The action principle will provide another level of compression: a scalar associated with a path will generate the equations of motion.