Graduate Classical Mechanics
Lecture 2: Discrete Mechanics—Chains, Rings, Forces, and Boundaries

From node values to edge differences and back again

1 From philosophy to a mechanical model

In Lecture 1 we introduced a working picture of dynamics as a sequence of complete states,

$$\cdots \longrightarrow S_{n-1} \longrightarrow S_n \longrightarrow S_{n+1} \longrightarrow \cdots , \tag{1}$$

with evolution generated by a rule

$$S_{n+1} = F(S_n). \tag{2}$$

We also emphasized a geometric distinction: state information naturally belongs to nodes, while a first difference naturally belongs to the edge joining two neighboring nodes.

In this lecture we make that idea mechanical. Our main example will be a one-dimensional chain of masses joined by springs. The chain is simple enough that every step can be written explicitly, but rich enough to introduce several ideas that will recur throughout the course:

- node variables and edge variables;
- incidence or difference matrices;
- force as a difference of edge tensions;
- second-order operators as compositions of first-order maps;
- the distinction between topology and constitutive or metric information;
- open boundaries versus periodic topology;
- fixed, free, forced, and mixed boundary conditions;
- discrete integration by parts;
- the origin of boundary terms;
- conservation of total internal force.

The point is not merely to obtain the familiar finite-difference form of a second derivative. The point is to understand what each operation does and where the resulting object lives.

2 A fixed–fixed chain with explicit boundaries

We begin with the boundary conditions visible from the start. Consider five nodes labeled

$$0, 1, 2, 3, 4, \tag{3}$$

where nodes 0 and 4 are fixed supports and nodes $1, 2, 3$ carry the three moving masses. Four springs connect adjacent nodes. Thus there are three dynamical displacement variables but four spring extensions.

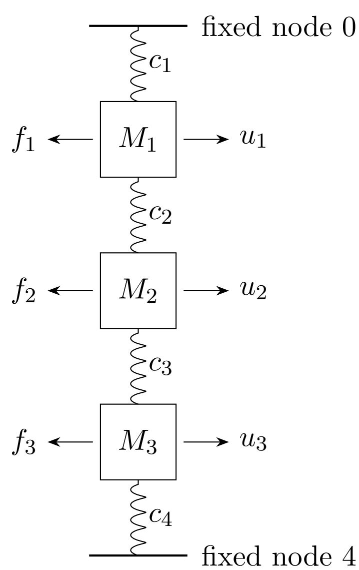
Let the displacement of the three moving masses from equilibrium be

$$\boxed{\boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}.} \tag{4}$$

The fixed supports have

$$u_0 = u_4 = 0. \tag{5}$$

A useful picture is



The four spring extensions from equilibrium are

$$\boldsymbol{e} = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 - u_1 \\ u_3 - u_2 \\ -u_3 \end{pmatrix}. \tag{6}$$

Therefore

$$\boxed{\boldsymbol{e} = A\boldsymbol{u},} \tag{7}$$

with

$$\boxed{A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}.} \tag{8}$$

The first and last rows are especially important. They are not artificial "boundary stencils" added after constructing an interior difference operator. They say directly that the first spring connects mass 1 to a fixed support and the fourth spring connects mass 3 to a fixed support.

Thus the shape and entries of A already contain the boundary information:

$$\boxed{\text{three free node variables} \xrightarrow{A} \text{four edge extensions.}} \tag{9}$$

For this fixed–fixed problem A has full column rank. In particular,

$$A\boldsymbol{u} = 0 \qquad \Longrightarrow \qquad \boldsymbol{u} = 0. \tag{10}$$

There is therefore no rigid-translation null mode. We will postpone the nullspace discussion until we consider systems, such as a free chain or a ring, in which translation is actually an allowed motion.

3 How the reduced matrix comes from the full chain

It is useful to see exactly how the fixed boundaries entered the matrix. Let

$$\widetilde{\boldsymbol{u}} = \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \tag{11}$$

contain the displacements of all five nodes before imposing the boundary conditions. The full incidence matrix is

$$\widetilde{A} = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}. \tag{12}$$

The fixed boundary conditions $u_0 = u_4 = 0$ mean that

$$\widetilde{\boldsymbol{u}} = B\boldsymbol{u}, \qquad B = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{13}$$

Hence

$$\boldsymbol{e} = \widetilde{A}\widetilde{\boldsymbol{u}} = \widetilde{A}B\boldsymbol{u}, \tag{14}$$

so the reduced matrix is simply

$$\boxed{A = \widetilde{A}B.} \tag{15}$$

This is an important model for what boundary conditions do: they restrict the allowed state space. Once that restriction is made, the resulting operator already knows about the boundary.

4 Potential energy and the stiffness/load operator

For linear springs, collect the spring constants into

$$C = \operatorname{diag}(k_{1/2}, k_{3/2}, \ldots, k_{N-1/2}). \tag{16}$$

The elastic potential energy relative to equilibrium is

$$\begin{aligned} U(\boldsymbol{u}) &= \frac{1}{2}\boldsymbol{e}^{\mathsf{T}} C\boldsymbol{e} && (17) \\ &= \frac{1}{2}(A\boldsymbol{u})^{\mathsf{T}} C(A\boldsymbol{u}) && (18) \\ &= \boxed{\frac{1}{2}\boldsymbol{u}^{\mathsf{T}} A^{\mathsf{T}} CA\boldsymbol{u}}. && (19) \end{aligned}$$

A variation gives

$$\begin{aligned} \delta U &= (A\boldsymbol{u})^{\mathsf{T}} C(A\,\delta\boldsymbol{u}) && (20) \\ &= (A^{\mathsf{T}} CA\boldsymbol{u})^{\mathsf{T}}\delta\boldsymbol{u}, && (21) \end{aligned}$$

so

$$\boxed{\nabla_{\boldsymbol{u}} U = A^{\mathsf{T}} C A \boldsymbol{u}.} \tag{22}$$

This is the positive stiffness or load operator. In static equilibrium, if $\boldsymbol{f}$ denotes the externally applied nodal load,

$$\boxed{A^{\mathsf{T}} C A \boldsymbol{u} = \boldsymbol{f}.} \tag{23}$$

This is the convention we will normally use in the course and is the familiar Strang form $A^{\mathsf{T}} C A u = f$.

The physical internal spring force on the nodes points downhill in potential energy:

$$\boxed{\boldsymbol{F}_{\text{int}} = -\nabla_{\boldsymbol{u}} U = -A^{\mathsf{T}} C A \boldsymbol{u}.} \tag{24}$$

Thus there is no contradiction between the two formulas. The quantity $A^{\mathsf{T}} C A \boldsymbol{u}$ is the external load required to hold the displacement $\boldsymbol{u}$ in static equilibrium; the springs exert the opposite internal force.

5 From edge extension to edge force and back to nodes

Collect the spring constants in

$$C = \operatorname{diag}(c_1, c_2, c_3, c_4). \tag{25}$$

The oriented edge-force vector is

$$\boxed{\boldsymbol{T} = C \boldsymbol{e} = C A \boldsymbol{u}.} \tag{26}$$

The load required at the three free masses is obtained by returning the edge forces to the nodes:

$$\boxed{\boldsymbol{f} = A^{\mathsf{T}} \boldsymbol{T} = A^{\mathsf{T}} C A \boldsymbol{u}.} \tag{27}$$

In components,

$$f_1 = T_1 - T_2, \tag{28}$$
$$f_2 = T_2 - T_3, \tag{29}$$
$$f_3 = T_3 - T_4. \tag{30}$$

The physical internal spring force is the opposite quantity,

$$\boxed{\boldsymbol{F}_{\text{int}} = -A^{\mathsf{T}} \boldsymbol{T} = -A^{\mathsf{T}} C A \boldsymbol{u}.} \tag{31}$$

The useful structural sequence is therefore

$$\boxed{\text{displacement on free nodes} \xrightarrow{A} \text{extension on edges} \xrightarrow{C} \text{edge force} \xrightarrow{A^\mathsf{T}} \text{required nodal load.}} \tag{32}$$

The first and last maps are determined by connectivity, orientation, and the choice of allowed node variables; the middle map contains the constitutive physics.

5.1 Why this is not the nilpotency statement

Later we will encounter the topological statement often written

$$d^2 = 0 \tag{33}$$

or, dually,

$$\partial^2 = 0. \tag{34}$$

The operator $A^\mathsf{T}CA$ is not an example of applying the same exterior derivative twice. The first operation maps node data to edge data. The return map is the adjoint/balance operation, together with constitutive information.

This distinction is worth maintaining from the beginning. A Laplacian-like operator is a composition of different geometrically meaningful maps; it is not simply "a derivative taken twice" in the topological sense.

6 The explicit fixed–fixed stiffness matrix

For the matrix above,

$$A^\mathsf{T}CA = \begin{pmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 + c_4 \end{pmatrix}. \tag{35}$$

For four identical springs, $C = cI$, so

$$\boxed{A^\mathsf{T}CA = c \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.} \tag{36}$$

Every free mass has two neighboring springs. The boundary is visible not through a weakened endpoint stencil, but through the fact that the first and last free masses are each connected to a fixed support.

Because A has full column rank and C is positive definite for ordinary springs,

$$\boldsymbol{u}^\mathsf{T} A^\mathsf{T} C A \boldsymbol{u} = (A\boldsymbol{u})^\mathsf{T} C (A\boldsymbol{u}) > 0 \tag{37}$$

for every nonzero $\boldsymbol{u}$. Thus the fixed–fixed stiffness matrix is positive definite.

7 Equations of motion

Let the node masses be collected in a diagonal mass matrix

$$M = \operatorname{diag}(m_1, m_2, m_3). \tag{38}$$

Because $\boldsymbol{q}^{(0)}$ is fixed, $\ddot{\boldsymbol{u}} = \ddot{\boldsymbol{q}}$. Newton's equation becomes

$$\boxed{M\ddot{\boldsymbol{u}} + A^\mathsf{T} C A \boldsymbol{u} = \boldsymbol{f},} \tag{39}$$

where $\boldsymbol{f}$ is the externally applied nodal force.

For identical masses m and springs c, the three free masses obey

$$m\ddot{\boldsymbol{u}} + c \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \boldsymbol{u} = \boldsymbol{f}. \tag{40}$$

In components,

$$m\ddot{u}_1 + c(2u_1 - u_2) = f_1, \tag{41}$$

$$m\ddot{u}_2 + c(2u_2 - u_1 - u_3) = f_2, \tag{42}$$

$$m\ddot{u}_3 + c(2u_3 - u_2) = f_3. \tag{43}$$

These equations resemble finite-difference equations with Dirichlet boundary values $u_0 = u_4 = 0$, but at present they are exact equations for this discrete mechanical model.

That distinction is central to the course philosophy:

> The discrete equation is not initially an approximation to the continuum equation. It is the exact equation of a different, finite mechanical system. A continuum theory may later emerge as a limit of a family of such systems.

8 The continuum analogue of $A^\mathsf{T} C A$

The matrix notation also anticipates the continuum operator. In one dimension, take

$$A = \frac{\mathrm{d}}{\mathrm{d}x}. \tag{44}$$

With the usual L^2 inner product, and with the boundary term kept separate, the formal adjoint is

$$A^{\mathsf{T}} = -\frac{\mathrm{d}}{\mathrm{d}x}. \tag{45}$$

Therefore

$$\boxed{A^{\mathsf{T}} C A u = -\frac{\mathrm{d}}{\mathrm{d}x}\left(C(x)\frac{\mathrm{d}u}{\mathrm{d}x}\right).} \tag{46}$$

The static equation $A^{\mathsf{T}} C A u = f$ becomes

$$\boxed{-\frac{\mathrm{d}}{\mathrm{d}x}\left(C(x)\frac{\mathrm{d}u}{\mathrm{d}x}\right) = f(x).} \tag{47}$$

Again, this is the external load required to balance the elastic response. The internal force density is its negative. This convention keeps $A^{\mathsf{T}} C A$ positive semidefinite in both the discrete and continuum descriptions.

9 Discrete integration by parts and explicit boundary terms

For the fixed–fixed problem, the boundary values have already been removed from the dynamical vector. To see a general boundary term explicitly, temporarily return to the full incidence matrix $\widetilde{A}$ acting on all $N + 1$ node values.

Let $\mathbf{1}_E$ denote the vector of ones in edge space, and define the point-supported node vectors $\mathbf{e}^{(j)}$ by

$$\boxed{(\mathbf{e}^{(j)})_i = \delta_{ij}.} \tag{48}$$

These are simply the standard basis vectors: each one is supported at a single node.

For an open chain oriented from left to right,

$$\boxed{\widetilde{A}^{\mathsf{T}} \mathbf{1}_E = -\mathbf{e}^{(0)} + \mathbf{e}^{(N)}.} \tag{49}$$

Therefore

$$\begin{aligned}
\mathbf{1}_E^{\mathsf{T}} \widetilde{A} \widetilde{\boldsymbol{u}} &= (\widetilde{A}^{\mathsf{T}} \mathbf{1}_E)^{\mathsf{T}} \widetilde{\boldsymbol{u}} && (50)\\
&= (-\mathbf{e}^{(0)} + \mathbf{e}^{(N)})^{\mathsf{T}} \widetilde{\boldsymbol{u}} && (51)\\
&= u_N - u_0. && (52)
\end{aligned}$$

This is the discrete fundamental theorem of calculus in linear-algebra form:

$$\boxed{\text{sum of edge differences} = \text{value on the oriented boundary.}} \tag{53}$$

The fixed–fixed matrix used earlier is $A = \widetilde{A}B$. Since the boundary components of $B\boldsymbol{u}$ vanish,

$$A^{\mathsf{T}}\mathbf{1}_E = B^{\mathsf{T}}\widetilde{A}^{\mathsf{T}}\mathbf{1}_E = 0. \tag{54}$$

So for the reduced fixed-boundary problem the boundary term has not been forgotten; it vanishes because the allowed displacement vector contains no boundary variation.

Now replace $\mathbf{1}_E$ by an arbitrary edge vector $\boldsymbol{v}$. The scalar identity

$$\boxed{\boldsymbol{v}^{\mathsf{T}}\widetilde{A}\widetilde{\boldsymbol{u}} = \widetilde{\boldsymbol{u}}^{\mathsf{T}}\widetilde{A}^{\mathsf{T}}\boldsymbol{v}} \tag{55}$$

is the discrete integration-by-parts identity in matrix form. Expanding it gives

$$\sum_{i=0}^{N-1} v_{i+1/2}(u_{i+1} - u_i) = -\sum_{i=1}^{N-1} u_i\left(v_{i+1/2} - v_{i-1/2}\right) \tag{56}$$

$$+ u_N v_{N-1/2} - u_0 v_{1/2}. \tag{57}$$

The matrix identity is the structural statement; the indexed expression shows the familiar split into interior balance and boundary terms.

This identity will recur in several disguises:

- variation of the action;
- derivation of Euler–Lagrange equations;
- field equations;
- flux conservation;
- self-adjoint differential operators;
- Green identities;
- generalized Stokes theorems.

The important habit is not to discard the boundary term automatically. It contains physical information.

10 Boundary conditions as different kinds of data

A differential equation is not a complete physical problem until the appropriate boundary or initial data have been specified. The discrete chain makes clear that boundary data can be of different geometric types.

10.1 Fixed boundary value

We may prescribe the displacement of a boundary node, for example

$$u_0 = 0. \tag{58}$$

In continuum language this becomes a Dirichlet-type condition.

10.2 Prescribed boundary force or tension

Alternatively, we may prescribe the edge force at the boundary. Schematically,

$$T_{1/2} = T_{\text{prescribed}}. \tag{59}$$

This is the discrete precursor of a Neumann or flux condition.

The two conditions are not merely different formulas. They specify different kinds of objects:

$$\boxed{\text{node value}} \qquad \text{versus} \qquad \boxed{\text{edge/flux value}}. \tag{60}$$

10.3 Mixed boundary condition

Suppose the endpoint is attached to an additional spring connected to a wall. Then the external restoring force is proportional to the endpoint displacement. Static force balance gives a relation between boundary displacement and boundary tension.

This is the discrete origin of a Robin or mixed boundary condition.

10.4 A useful subtlety: a boundary mass is not a boundary condition

If the endpoint node carries a finite mass and is allowed to move, then it is an ordinary dynamical degree of freedom. Its equation is, for example,

$$m_0 \ddot{u}_0 = T_{1/2} + F_0^{\text{ext}}. \tag{61}$$

That is not the same as imposing zero tension at the endpoint.

A “free end” condition such as zero boundary tension is appropriate when the field endpoint itself has no independent boundary inertia or when the variational problem leaves the boundary value free and no external boundary force is applied.

This distinction becomes important when passing between particle models and continuum field models.

11 Boundary conditions change the state space and the operator

The initial three-mass example has fixed endpoint values built in from the beginning. The dynamical state is

$$\boldsymbol{u} = (u_1, u_2, u_3)^\mathsf{T}, \tag{62}$$

not the five-component vector that would include u_0 and u_4. Consequently the 4×3 matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \tag{63}$$

already represents the fixed boundary conditions.

A different physical boundary condition changes either the allowed state variables, the incidence map, or both. This is why boundary conditions are not merely instructions appended after writing a bulk differential equation. They help define the operator itself.

12 Closing the chain into a ring

Now identify the final node with the first after one circuit and connect every node to two neighbors. With N nodes there are now also N edges.

For a four-node ring, choose orientations

$$0 \to 1, \qquad 1 \to 2, \qquad 2 \to 3, \qquad 3 \to 0. \tag{64}$$

Then

$$A_{\text{ring}} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix}. \tag{65}$$

Therefore

$$A_{\text{ring}}^{\mathsf{T}} A_{\text{ring}} = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}. \tag{66}$$

There are no special endpoint rows because there are no endpoints. Unlike the fixed–fixed chain, the ring permits a rigid translation:

$$\boxed{A_{\text{ring}} \mathbf{1} = 0.} \tag{67}$$

Thus the uniform vector belongs to the nullspace of A_{ring} and of $A_{\text{ring}}^{\mathsf{T}} C A_{\text{ring}}$. This is the natural place to introduce the translational zero mode: it appears because the boundary conditions allow it.

The difference between the fixed chain and the ring is not local spring physics. Locally the same constitutive law can apply. The distinction is global topology and boundary structure.

13 Closed loops and the fundamental theorem

For the ring,

$$\sum_{i=0}^{N-1} (u_{i+1} - u_i) = 0, \tag{68}$$

where $q_N \equiv q_0$.

This is the closed-loop version of the telescoping identity from Lecture 1.

For an open chain,

$$\sum \Delta q = q_{\text{right}} - q_{\text{left}}. \tag{69}$$

For a closed ring,

$$\sum_{\text{loop}} \Delta q = 0. \tag{70}$$

These are the same statement:

> The accumulated exact difference is determined by the boundary. A closed loop has no boundary.

This is the one-dimensional seed of generalized Stokes.

14 General edge data need not be a gradient

Suppose we assign arbitrary values $a_{i+1/2}$ to the edges of a ring. Must there exist node values u_i such that

$$a_{i+1/2} = u_{i+1} - u_i? \tag{71}$$

No. A necessary condition is

$$\boxed{\sum_{\text{ring}} a_{i+1/2} = 0.} \tag{72}$$

An arbitrary edge field can have nonzero circulation, whereas an exact difference of a single-valued node field cannot.

This is our first simple example of a local object containing global information. Later, in more advanced language, this becomes a distinction among general, closed, and exact differential forms. We do not need that terminology yet; the ring already shows the essential idea.

For an angular variable, there is an important modification. If

$$\theta_N = \theta_0 + 2\pi m, \tag{73}$$

then

$$\sum_i \Delta\theta_{i+1/2} = 2\pi m. \tag{74}$$

The integer m is a winding number. This is a useful reminder that global topology can survive even when local differences look perfectly ordinary.

15 Internal forces, support reactions, and conservation

For a system containing only internal pair forces and no fixed supports, incidence structure makes Newton's third-law cancellation transparent. For the ring,

$$\mathbf{1}^\mathsf{T}\boldsymbol{F}_{\text{int}} = -\mathbf{1}^\mathsf{T} A_{\text{ring}}^\mathsf{T}\boldsymbol{T} = -(A_{\text{ring}}\mathbf{1})^\mathsf{T}\boldsymbol{T} = 0. \tag{75}$$

Thus

$$\boxed{\sum_i F_i^{\text{int}} = 0} \tag{76}$$

for the complete ring.

For the fixed–fixed three-mass system, however,

$$A\mathbf{1} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \neq 0. \tag{77}$$

This does *not* mean Newton's third law has failed. It means the reduced system contains the three moving masses but not the two fixed supports. The springs attached to the walls exert external forces on that reduced subsystem. If the support reactions are included, total internal force again cancels.

This is another reason to keep the boundary explicit: whether a force is "internal" depends on what has been included in the state being modeled.

16 Energy and positive definiteness versus semidefiniteness

For the fixed–fixed chain,

$$U = \frac{1}{2}(A\boldsymbol{u})^\top C(A\boldsymbol{u}), \tag{78}$$

so, for positive spring constants,

$$U > 0 \qquad \text{for every } \boldsymbol{u} \neq 0. \tag{79}$$

Therefore

$$\boxed{A^\top CA \text{ is symmetric positive definite}} \tag{80}$$

for this fixed–fixed model.

For the ring,

$$A_{\text{ring}}\mathbf{1} = 0, \tag{81}$$

so

$$A_{\text{ring}}^\top CA_{\text{ring}}\mathbf{1} = 0. \tag{82}$$

The ring stiffness matrix is therefore positive semidefinite and has a translational zero mode.

This comparison makes an important point: boundary conditions change not only the solutions but also the nullspace and definiteness of the operator.

17 A short computational experiment: a static boundary-value problem

A useful first computational experiment is to solve a larger fixed–fixed chain under a constant applied load. This keeps the focus on the static equation

$$\boxed{A^{\mathsf{T}} C A \boldsymbol{u} = \boldsymbol{f}} \tag{83}$$

before we introduce normal modes and eigenvalues later in the course.

Take M free masses between two fixed endpoints. There are therefore $M+2$ nodes and $M+1$ springs. With the same orientation used above, the reduced incidence matrix has size

$$A \in \mathbb{R}^{(M+1)\times M}. \tag{84}$$

For unit spring constants, $C = I$, and for a constant unit load,

$$\boldsymbol{f} = \mathbf{1}. \tag{85}$$

The static displacement of the free masses is therefore found from

$$\boldsymbol{u}_{\text{free}} = (A^{\mathsf{T}} C A)^{-1} \mathbf{1}, \tag{86}$$

or, computationally, by solving the linear system rather than explicitly forming the inverse.

For unit spacing the equation at an interior node is

$$2u_i - u_{i-1} - u_{i+1} = 1, \tag{87}$$

with fixed-boundary conditions

$$u_0 = u_{M+1} = 0. \tag{88}$$

The quadratic

$$\boxed{u_i = \frac{i(M+1-i)}{2}} \tag{89}$$

satisfies this discrete equation exactly. Thus the same parabola that solves the continuum problem

$$-u''(x) = 1, \qquad u(0) = u(M+1) = 0, \tag{90}$$

also passes exactly through the discrete solution at the nodes. This makes the example a particularly clean bridge between the discrete and continuum descriptions.

MATLAB version

The following is a convenient in-class demonstration. MATLAB's `toeplitz` command constructs the incidence matrix directly.

```
% Number of free masses
M = 3;

% Number of springs. There are M+2 nodes including the two
% fixed endpoints, so there are M+1 springs.
N = M + 1;

% Reduced incidence/difference matrix, size (M+1)-by-M.
% The first row measures extension from the fixed left endpoint
% to mass 1; the last row measures extension from mass M to the
% fixed right endpoint.
A = toeplitz([1 -1 zeros(1,N-2)], ...
            [1 zeros(1,M-1)]);

% Node positions, including the two fixed endpoints.
q = 0:M+1;

% Constant unit load on every free mass.
f = ones(M,1);

% Unit spring constants.
C = eye(N,N);

% Solve A'*C*A*u_free = f and then restore the fixed endpoints.
% The backslash solves the linear system; it does not form an inverse.
u = [0; (A'*C*A)\f; 0];

% Exact quadratic solution at the node locations.
u_exact = -q.*(q-(M+1))/2;

% Compare the discrete solution with the quadratic.
plot(q,u,'o',q,u_exact,'-');
xlabel('node position');
ylabel('displacement');
legend('discrete solution','quadratic solution','Location','best');
grid on;
```

The two curves lie on top of one another at the nodes. This is stronger than merely observing convergence as the mesh is refined: for this constant-coefficient, constant-load problem, the quadratic has exactly the constant second difference required by the discrete equations.

Python/NumPy version

The same experiment can be written using NumPy arrays. Here we construct A directly so that the correspondence with the MATLAB matrix remains visible.

```
import numpy as np
```

```
import matplotlib.pyplot as plt

# Number of free masses
M = 3

# Number of springs
N = M + 1

# Reduced incidence/difference matrix, shape (M+1, M)
A = np.zeros((N, M))
j = np.arange(M)
A[j, j] = 1.0
A[j + 1, j] = -1.0

# Node positions, including the two fixed endpoints
q = np.arange(M + 2, dtype=float)

# Constant unit load and unit spring constants
f = np.ones(M)
C = np.eye(N)

# Solve A.T @ C @ A @ u_free = f
u_free = np.linalg.solve(A.T @ C @ A, f)

# Restore the two fixed endpoint values
u = np.concatenate((np.zeros(1), u_free, np.zeros(1)))

# Exact quadratic solution at the node locations
u_exact = -q * (q - (M + 1)) / 2.0

# Compare the discrete and quadratic solutions
plt.plot(q, u, 'o', label='discrete solution')
plt.plot(q, u_exact, '-', label='quadratic solution')
plt.xlabel('node position')
plt.ylabel('displacement')
plt.legend()
plt.grid(True)
plt.show()
```

Questions worth asking during or after the computation are:

1. Why is A rectangular, with one more row than column?
2. Where are the two fixed-boundary conditions encoded in A?
3. Why does the linear system have a unique solution in this fixed–fixed case?
4. Why does a quadratic produce a constant second difference?
5. What changes if the entries of C are not all equal?
6. What changes if the applied load $\boldsymbol{f}$ is not constant?

18 What the continuum notation will eventually compress

At an interior point of a uniform chain,

$$F_i^{\text{int}} = k(u_{i+1} - 2u_i + u_{i-1}). \tag{91}$$

If the nodes are separated by a spacing a, then

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{a^2} \tag{92}$$

will eventually approach a second spatial derivative under an appropriate continuum limit.

But by the time we write

$$\frac{\partial^2 u}{\partial x^2}, \tag{93}$$

we should remember what has been compressed into the notation:

- field values live on one kind of geometric object;
- first differences live on another;
- constitutive physics acts on the first differences;
- force or flux balance returns information to the original type of object;
- boundaries require their own specification;
- closed topology changes global consistency conditions.

The continuum operator is powerful precisely because it compresses this structure. Our goal is to use the compression without forgetting what it contains.

19 Summary

The main sequence of this lecture is

$$\boxed{\boldsymbol{u} \xrightarrow{A} \boldsymbol{e} \xrightarrow{C} \boldsymbol{T} \xrightarrow{A^\mathsf{T}} \boldsymbol{f}.} \tag{94}$$

For a spring system in static equilibrium,

$$\boxed{\boldsymbol{f} = A^\mathsf{T} C A \boldsymbol{u},} \tag{95}$$

while the internal spring force is

$$\boxed{\boldsymbol{F}_{\text{int}} = -A^{\mathsf{T}} C A \boldsymbol{u}.} \tag{96}$$

The matrix A records connectivity and orientation. The matrix C records constitutive physics. Their composition is the positive stiffness/load operator on node displacements.

An open chain has a boundary, and boundary terms survive discrete integration by parts. A ring has no boundary, so exact differences sum to zero around the loop. Boundary conditions can prescribe node values, edge or flux values, or relations between them.

The broader lesson is:

> *Do not let compact notation hide where quantities live, how operators are assembled, or what information belongs to the boundary.*

Questions to keep in mind

1. Which parts of $A^{\mathsf{T}} CA$ depend only on connectivity, and which depend on the physics of the springs?
2. Why is the fixed–fixed matrix A full column rank, while A_{ring} has the uniform vector in its nullspace?
3. What changes algebraically when an open chain is closed into a ring?
4. In what sense is a boundary condition part of the definition of the physical problem rather than merely an extra equation?
5. Why is $A^{\mathsf{T}} A$ not an example of the statement $d^2 = 0$?
6. What information is contained in the boundary terms of the discrete integration-by-parts identity?
7. Under what condition can edge data on a ring be written as differences of a globally defined node variable?
8. How does internal-force cancellation anticipate momentum conservation and Noether's theorem?