Graduate Classical Mechanics

Lecture 1: States, Dynamics, and the Goal of Physics

Course framing and conceptual foundations

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1 Why begin a mechanics course this way?

A graduate mechanics course is often organized as a sequence of techniques: generalized coordinates, the calculus of variations, Lagrange's equations, central forces, rigid bodies, Hamiltonian mechanics, canonical transformations, and so on. Those subjects are important, but they can look like a collection of increasingly sophisticated methods unless we first ask a more basic question:

> What are we trying to accomplish when we do physics?

In this course we will use a simple working picture. At some stage the universe, or the physical system under consideration, is in a state S_n. At the next stage it is in a state S_{n+1}:

$$\cdots \longrightarrow S_{n-1} \longrightarrow S_n \longrightarrow S_{n+1} \longrightarrow S_{n+2} \longrightarrow \cdots . \tag{1}$$

The symbol S_n is deliberately general. It represents everything that is required, at that stage, to determine the subsequent state. Our working assumption will be that there exists a rule

$$\boxed{S_{n+1} = F(S_n)} \tag{2}$$

for the complete state.

This statement is the conceptual starting point of the course. It says that the dynamics closes on the complete state.

The notation does *not* yet assume that the underlying world is fundamentally discrete. The labels n may denote genuinely discrete stages, or they may denote samples of a continuous history. If continuous differential equations are fundamental, their exact finite-time flow still defines a map from one complete state to the next sampled state. We will make this correspondence explicit when we return to Newtonian mechanics in Lecture 3.

For many systems of interest, the evolution is also reversible, so that F is invertible and

$$S_n = F^{-1}(S_{n+1}). \tag{3}$$

We will not need to assume invertibility in every calculation, but it is useful to distinguish the existence of a unique next state from the stronger statement that the previous state is also uniquely recoverable.

2 History versus law

One possible way to describe the history of a system is simply to list every state:

$$S_0, \quad S_1, \quad S_2, \quad S_3, \quad \ldots \tag{4}$$

Such a list may be complete, but it provides very little insight. It tells us what happened, not why neighboring states are related in the way they are.

A physical law provides an enormous compression of this information. If the rule F is known, then the complete history can be generated from an initial state:

$$S_1 = F(S_0), \tag{5}$$
$$S_2 = F(S_1) = F(F(S_0)), \tag{6}$$
$$S_3 = F(S_2), \tag{7}$$
$$\vdots \tag{8}$$

Thus an enormous list of states can be replaced by

$$\boxed{\text{initial state} + \text{dynamical rule.}} \tag{9}$$

This is one of the remarkable facts about physics. Nature appears to admit compact rules that generate an enormous variety of possible histories.

A useful way to state the philosophy is:

> *The universe gives us histories. Physics searches for the states, constraints, and compact rules that generate them.*

The compression is not merely aesthetic. It makes prediction, explanation, engineering, and technology possible. We do not have to store the future in order to predict it; we need a sufficiently complete state and a sufficiently accurate rule.

3 The first difficult problem: what is the state?

The equation

$$S_{n+1} = F(S_n) \tag{10}$$

looks simple, but it hides one of the central problems of physics:

> What information must be included in S_n?

Suppose we try to describe a classical particle using only its position,

$$S_n = q_n. \tag{11}$$

In general this is not enough. Two particles can pass through the same position with different velocities and have different futures. The evolution does not close on q alone.

We therefore enlarge the state. In elementary mechanics a natural choice is

$$S = (q, v), \tag{12}$$

or, in Hamiltonian language,

$$S = (q, p). \tag{13}$$

Then, given the equations of motion, the next state is determined.

This gives us a useful working definition:

A state description is dynamically complete when the evolution closes on that state.

If a proposed set of variables does not determine its own future, one possibility is that the dynamical law is genuinely nonlocal in those variables. Another, often more useful possibility is that we have omitted variables that are part of the complete state.

This simple observation is already related to the order of differential equations. A second-order equation for $q(t)$ tells us that position at one instant is not by itself a complete state. We require enough information to specify both q and its first change. Hamiltonian mechanics makes this explicit by enlarging configuration space to phase space and replacing a second-order equation by a system of first-order equations.

4 Complete states and projected descriptions

In practice we almost never manipulate the complete state of everything relevant to a process. Instead we study a subsystem, an observable, or a reduced set of variables.

Let the complete state be S, and suppose our description retains only

$$\boxed{s = \mathsf{P}(S).} \tag{14}$$

Here P is a projection in a broad sense: it means that we keep some information and ignore the rest.

The complete state evolves as

$$S_{n+1} = F(S_n). \tag{15}$$

The reduced description therefore evolves according to

$$s_{n+1} = \mathsf{P}(F(S_n)). \tag{16}$$

There need not exist a closed function f such that

$$s_{n+1} = f(s_n). \tag{17}$$

Indeed, two different complete states may have the same reduced description,

$$\mathsf{P}(S) = \mathsf{P}(S'), \tag{18}$$

while their reduced futures differ,

$$\mathsf{P}(F(S)) \neq \mathsf{P}(F(S')). \tag{19}$$

The obstruction can be summarized schematically as

$$\boxed{\mathsf{P} \circ F \neq f \circ \mathsf{P}} \tag{20}$$

for any autonomous reduced map f.

This is a general source of effective complexity. Apparent randomness, dissipation, memory, and irreversibility can arise because the variables we retain do not form a complete state.

4.1 A classical example: dissipation

Consider a pendulum moving in air. If we describe only the pendulum coordinate, we may write an effective damped equation such as

$$ml^2\ddot{\theta} + b\dot{\theta} + mgl\sin\theta = 0. \tag{21}$$

Mechanical energy of the pendulum decreases. But this does not require the complete microscopic state to obey a fundamentally dissipative law. Energy is transferred into degrees of freedom we chose not to track: motion of the surrounding air, internal molecular motion, sound, thermal energy, and so on.

Schematically,

$$S = (S_{\text{pendulum}}, S_{\text{environment}}) \tag{22}$$

may evolve under a closed rule F, while

$$s = \mathsf{P}(S) = S_{\text{pendulum}} \tag{23}$$

obeys an effective dissipative equation.

This distinction will reappear in many areas of physics: coarse graining, statistical mechanics, effective field theory, open quantum systems, and reduced-order models.

5 States live on nodes; changes live on transitions

The state sequence suggests a geometric picture:

$$\boxed{S_n} \quad \xrightarrow{\Delta S_{n+1/2}} \quad \boxed{S_{n+1}}. \tag{24}$$

Define the finite change

$$\boxed{\Delta S_{n+1/2} = S_{n+1} - S_n.} \tag{25}$$

The half-integer index is deliberate. It reminds us that S_n belongs to a node while the first difference belongs to the edge connecting two nodes.

The relation

$$S_{n+1} = S_n + \Delta S_{n+1/2} \tag{26}$$

makes something elementary but important explicit:

> A change is not itself a state. It is information about the transition between states.

Knowing $\Delta S_{n+1/2}$ without knowing S_n does not tell us S_{n+1}.

For example, knowing that a particle moved one meter does not tell us its final position unless we also know its initial position.

This distinction between objects that live on nodes and objects that live on transitions will recur throughout the course.

6 Finite differences first

We will generally begin with finite differences when the discrete structure is natural. This is not a rejection of derivatives, infinitesimals, or continuum mathematics. Rather, finite differences often make the geometry of the construction more transparent.

We begin with

$$\Delta q_{n+1/2} = q_{n+1} - q_n. \tag{27}$$

If the states are separated by a time interval h, we can form

$$\frac{\Delta q_{n+1/2}}{h} = \frac{q_{n+1} - q_n}{h}. \tag{28}$$

Only later do we take an appropriate limit,

$$\frac{q_{n+1} - q_n}{h} \longrightarrow \frac{\mathrm{d}q}{\mathrm{d}t}. \tag{29}$$

The derivative is then understood as a compact continuum representation of a relation between neighboring states.

7 The fundamental theorem as telescoping and boundary

Finite differences make the one-dimensional fundamental theorem almost unavoidable.

For a chain of states from M to N,

$$\sum_{n=M}^{N-1} \Delta S_{n+1/2} = \sum_{n=M}^{N-1} (S_{n+1} - S_n) \tag{30}$$

$$= (S_{M+1} - S_M) + (S_{M+2} - S_{M+1}) + \cdots + (S_N - S_{N-1}) \tag{31}$$

$$= S_N - S_M. \tag{32}$$

Thus

$$\boxed{\sum_{n=M}^{N-1} \Delta S_{n+1/2} = S_N - S_M.} \tag{33}$$

Every contribution from an interior node cancels. Only the boundary survives.

The continuum version is

$$\boxed{\int_{t_M}^{t_N} \mathrm{d}S = S(t_N) - S(t_M).} \tag{34}$$

We will later recognize this as the one-dimensional member of a much larger family of statements summarized by the generalized Stokes theorem:

$$\boxed{\int_{\Omega} \mathrm{d}\omega = \int_{\partial\Omega} \omega.} \tag{35}$$

For now, the important idea is simply

$$\boxed{\text{local changes in the interior accumulate to boundary information.}} \tag{36}$$

8 Open chains, closed rings, and the visibility of boundaries

Suppose we have $N+1$ nodes arranged in an open chain. There are N connecting edges. The telescoping sum leaves the two endpoints.

Now close the chain into a ring. Let $S_N \equiv S_0$. Then

$$\sum_{n=0}^{N-1} (S_{n+1} - S_n) = 0. \tag{37}$$

The closed loop has no boundary, so an exact difference sums to zero around it.

This example is useful because it shows that the familiar "one fewer edge than node" rule is not fundamental. It is a property of an interval. A ring with N nodes has N edges.

The deeper statement is that quantities live on different geometric objects, and topology determines how those objects fit together.

This will later connect to circulation, periodic boundary conditions, Fourier modes, winding, and fields on closed spaces.

9 Second differences and why location matters

To construct a centered second difference, start with the two edge differences adjacent to node n:

$$\Delta S_{n+1/2} = S_{n+1} - S_n, \tag{38}$$

$$\Delta S_{n-1/2} = S_n - S_{n-1}. \tag{39}$$

Their difference is

$$\Delta^2 S_n = \Delta S_{n+1/2} - \Delta S_{n-1/2} \tag{40}$$

$$= S_{n+1} - 2S_n + S_{n-1}. \tag{41}$$

The first difference is associated with edges. The second difference returns to a node-centered quantity, but it requires information from neighboring states.

Schematically,

$$\boxed{\text{nodes}} \xrightarrow{D} \boxed{\text{edges}} \xrightarrow{D^*} \boxed{\text{nodes}}. \tag{42}$$

Later we will see this structure physically in a spring chain:

$$\boxed{\text{displacement on nodes}} \longrightarrow \boxed{\text{extension on edges}} \longrightarrow \boxed{\text{tension on edges}} \longrightarrow \boxed{\text{force on nodes}}. \tag{43}$$

This is more informative than saying only that a second derivative is "the derivative of a derivative."

10 Boundary conditions are part of the physics

Once we keep track of where quantities live, boundary conditions become more concrete.

A second-order equation generally requires additional information at a boundary or at an initial time. In a discrete system, we can see explicitly why: interior nodes interact with objects on both sides, while a boundary node does not.

Different physical boundary conditions may specify different types of data.

For example, in a discrete string we may specify the displacement of an endpoint, which is node data, or the tension at the endpoint, which is edge-like or flux-like data.

In continuum language these become familiar alternatives such as

$$\phi|_{\partial\Omega} = \text{given} \tag{44}$$

or

$$\boldsymbol{n} \cdot \nabla\phi|_{\partial\Omega} = \text{given}. \tag{45}$$

We will introduce the standard names later. The conceptual point comes first:

> *Boundary conditions are not decorations attached to a differential equation. They are part of the specification of the physical problem.*

11 The boundary of a boundary

The cancellation in the telescoping sum is the simplest example of another principle that will recur:

$$\boxed{\partial^2 = 0.} \tag{46}$$

The boundary of an oriented edge consists of its two endpoints with opposite signs. If edges are assembled into a closed loop, the endpoint contributions cancel. In higher dimensions, interior edges cancel when oriented faces are assembled, and interior faces cancel when volumes are assembled.

We do not need the formal machinery of algebraic topology or differential forms at this point. The important structural lesson is that generalized Stokes is already visible in finite combinatorics.

12 Two notions of time

The state sequence

$$S_0 \to S_1 \to S_2 \to \cdots \tag{47}$$

raises a subtle question: what do we mean by time?

There are at least two concepts that should be distinguished.

First, there is the ordering of the states. The index n tells us that one state follows another.

Second, there is operational time: the quantity measured by physical clocks.

These concepts need not be identified at the outset.

12.1 A simulation analogy

Imagine that the state sequence is generated by a computer simulation. The external computer could execute the update rule rapidly or slowly. Changing the processor speed need not change the sequence

$$S_0, S_1, S_2, \ldots. \tag{48}$$

The wall-clock time of the computer is not automatically a physical observable for an observer represented inside the simulated states.

How could such an observer measure time? By using some subsystem whose evolution is sufficiently regular as a clock. If a subsystem exhibits a repeating pattern, other changes can be compared with it.

Operationally, then, a clock is a physical subsystem used to label change in other physical subsystems.

This motivates the distinction

ordering of states	versus	time measured by physical clocks.

(49)

The distinction becomes especially important in relativity and in formulations of gravitation where there is no universal external time parameter.

We will not attempt to settle the ontology of time in this course. The point is simply that the label used to order states and the time read by a physical clock are conceptually distinguishable.

13 Quantum mechanics as an example of state evolution

Although this is a classical mechanics course, quantum theory provides a useful example of the general state-evolution picture.

For a closed quantum system, the state vector evolves unitarily. Schematically,

$$\boxed{|\Psi_{n+1}\rangle = U_n\,|\Psi_n\rangle\,.} \tag{50}$$

This has precisely the form

$$S_{n+1} = F(S_n). \tag{51}$$

Unitary evolution is invertible, so the complete quantum state has both a unique successor and a unique predecessor.

13.1 Measurement and projected descriptions

Suppose a two-state system interacts with a measuring apparatus. A schematic unitary interaction is

$$(a\,|0\rangle + b\,|1\rangle)\,|A_{\text{ready}}\rangle \longrightarrow a\,|0\rangle\,|A_0\rangle + b\,|1\rangle\,|A_1\rangle\,. \tag{52}$$

The total state evolves according to the same unitary rule. What an observer calls a particular measurement outcome concerns a restricted, correlated part of the total state.

One can therefore apply the same complete-state/reduced-description distinction:

$$|\Psi\rangle \longrightarrow \mathsf{P}(|\Psi\rangle). \tag{53}$$

From an Everett-like point of view, the apparent selection of an outcome is associated with the branch of the global state containing the observer's correlated record, rather than with a fundamental change in the universal evolution law.

Other interpretations of quantum mechanics assign a different physical meaning to the same formalism, and objective-collapse models explicitly modify the unitary dynamics. These are foundational questions rather than subjects we need to resolve here.

For the purposes of this course, the important lesson is narrower:

> *One should distinguish the evolution of a complete state from the effective evolution of the limited variables an observer chooses or is able to examine.*

This is the same conceptual distinction that appears in ordinary dissipative classical mechanics.

14 General relativity and hypersurface states

General relativity is structurally more subtle because spacetime geometry itself is dynamical. Nevertheless, canonical formulations again organize the theory in terms of data associated with a spatial hypersurface, constraints on those data, and equations governing their evolution.

In the ADM formulation, for example, one decomposes spacetime into spatial slices. The spatial geometry and its conjugate data play a role analogous to generalized coordinates and momenta, while constraint equations restrict the allowed states.

The details are far beyond what we need here. The point is simply that a broad range of modern physics can be organized conceptually in terms of

$$\boxed{\text{state space} + \text{constraints} + \text{dynamics.}} \tag{54}$$

Classical mechanics is where we can study this architecture in its cleanest form.

15 Different formulations expose different structures

The basic problem of mechanics can be stated simply:

$$\boxed{\text{Given the current state, determine the next state.}} \tag{55}$$

Newtonian, Lagrangian, and Hamiltonian mechanics are different ways of representing the dynamical rule.

In Newtonian mechanics we often write

$$m\ddot{q} = F(q, \dot{q}, t). \tag{56}$$

In Lagrangian mechanics we encode the dynamics using

$$L(q, \dot{q}, t) \tag{57}$$

and obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0. \tag{58}$$

In Hamiltonian mechanics we use the state

$$S = (q, p) \tag{59}$$

and the first-order system

$$\dot{q} = \frac{\partial H}{\partial p}, \tag{60}$$

$$\dot{p} = -\frac{\partial H}{\partial q}. \tag{61}$$

These are not simply three different algorithms for obtaining trajectories. Each representation makes different structure visible.

Newtonian mechanics emphasizes forces and acceleration.

Lagrangian mechanics emphasizes configuration space, action, constraints, and symmetry.

Hamiltonian mechanics emphasizes complete state space, first-order evolution, phase-space geometry, conserved structures, and transformations of the state variables.

One of the central goals of this course is to understand why changing representation can reveal physics that was difficult to see before.

16 A preview of the discrete action viewpoint

The state-and-transition picture also suggests how we will introduce variational mechanics.

Instead of beginning with a smooth function $q(t)$, imagine a discrete path

$$q_0, q_1, \ldots, q_N. \tag{62}$$

A discrete action is a function of the entire finite path,

$$S_d[q] = \sum_{n=0}^{N-1} L_d(q_n, q_{n+1}). \tag{63}$$

We can vary the node values using ordinary finite-dimensional calculus. An interior node appears in neighboring terms, and stationarity produces a discrete equation of motion.

Only afterward do we take a continuum limit and obtain the usual action

$$S[q] = \int L(q, \dot{q}, t)\, \mathrm{d}t. \tag{64}$$

This order will let us see boundary terms, neighboring-state structure, and the origin of the Euler–Lagrange equation before those ideas are compressed into continuum notation.

17 What should we carry forward?

The philosophical framing of this lecture can be summarized in a small number of statements.

1. A physical history can be represented as an ordered sequence of states.
2. Our working assumption is that a complete state evolves according to
$$S_{n+1} = F(S_n).$$
3. Finding the correct state variables is part of finding the theory.
4. Reduced descriptions $s = \mathsf{P}(S)$ need not have closed, reversible, or deterministic-looking dynamics even when the complete state does.
5. States naturally live on nodes and first changes naturally live on transitions or edges.
6. Finite differences make boundary structure explicit.
7. The fundamental theorem of calculus is the simplest example of the principle that accumulated local change becomes boundary information.
8. Boundary conditions are part of the physics, not an afterthought.
9. Physical laws compress enormous families of possible histories into compact rules.
10. Newtonian, Lagrangian, and Hamiltonian mechanics are different representations of those rules, each exposing different structure.

Questions to keep in mind

As we proceed through the course, we will repeatedly return to questions like these:

1. What exactly is the state of the system?
2. Does the dynamics close on the variables we have chosen?
3. What information has been projected out of an effective description?
4. Which quantities belong to states, and which belong to transitions?
5. Where do the relevant mathematical objects live geometrically?
6. What is the boundary of the system, and what data are specified there?
7. Which structures belong to the physical system, and which arise from our choice of coordinates?
8. What does a given formulation of mechanics make easy to see?
9. What does a physical clock actually measure?
10. How much of a physical history can be compressed into a useful law?

Guiding idea: The universe gives us histories. Physics searches for the states, constraints, and compact rules that generate them. Mechanics is our first systematic study of how to represent those rules and expose the structures they preserve.