

Lecture 14 Instructor Outline — Draft

Rigid Bodies II: Dynamics, Symmetry, and the Heavy Top

PHYS V1100: Analytical Dynamics — 105 minutes

Euler equations, stability, symmetry reduction, and orientation representations

Core arc:

$$\begin{aligned} \dot{\mathbf{J}}_s = \boldsymbol{\tau}_s \rightarrow \boldsymbol{\tau}_b = \dot{\mathbf{J}}_b + \boldsymbol{\omega}_b \times \mathbf{J}_b \rightarrow \mathbf{l}_b \dot{\boldsymbol{\omega}}_b + \boldsymbol{\omega}_b \times (\mathbf{l}_b \boldsymbol{\omega}_b) = \boldsymbol{\tau}_b \\ \Downarrow \\ \text{principal axes} \rightarrow \text{torque-free invariants} \rightarrow \text{stability} \rightarrow \text{symmetric top} \\ \Downarrow \quad \text{beyond Exam II cutoff} \\ Q \rightarrow Q = Q_{\mathbf{n}} R_3(\chi) \rightarrow (\mathbf{n}, \text{spin}) \rightarrow \text{heavy-top reduction and quadrature} \end{aligned}$$

Protected goals.

- Begin from the rotating-frame derivative established in Lecture 13; do not rederive rigid-body kinematics.
- Derive Euler's vector equation before choosing principal axes, then obtain the component Euler equations as a basis choice.
- Use torque-free conservation and linearization to explain principal-axis stability and the intermediate-axis instability.
- Solve the torque-free symmetric top in body components and interpret its space-frame precession about fixed angular momentum.
- Mark the Exam II rigid-body boundary after the torque-free symmetric top; PSet6 Q15–Q16 cover this required scope.
- After the cutoff, compare orientation representations: Q as the direct frame map, a rotor as a compact geometric representation, and Euler angles as one local decomposition/coordinate choice for Q .
- Use symmetry to motivate $Q = Q_{\mathbf{n}} R_3(\chi)$ for the heavy symmetric top, and derive the standard one-dimensional nutation quadrature without requiring a full three-angle Euler treatment.

Notation and scope. Subscripts s, b denote space/laboratory and body components of the same physical vector. The body inertia tensor $\mathbf{l}_b$ is constant. In the heavy-top section the pivot is fixed, $\mathbf{n} = Q\mathbf{e}_3$ is the space direction of the body symmetry axis, and $\mathbf{e}_z$ is the upward laboratory vertical. The gravitational potential is therefore $V = Mg\ell \mathbf{e}_z \cdot \mathbf{n}$.

0–12 min: Angular-momentum balance in a moving body frame

BOARD Start exactly where Lecture 13 stopped:

$$\boxed{\dot{\mathbf{J}}_s = \boldsymbol{\tau}_s,} \quad \boxed{Q^T \dot{\mathbf{J}}_s = \dot{\mathbf{J}}_b + \boldsymbol{\omega}_b \times \mathbf{J}_b.}$$

Therefore

$$\boxed{\boldsymbol{\tau}_b = \dot{\mathbf{J}}_b + \boldsymbol{\omega}_b \times \mathbf{J}_b.}$$

SAY In the inertial frame the balance law is simple. In the body frame the mass distribution and inertia tensor are simple. The cross-product term is the price of using both advantages at once.

BOARD Since Lecture 13 gave

$$\boxed{\mathbf{J}_b = \mathbf{l}_b \boldsymbol{\omega}_b}$$

and $\mathbf{l}_b$ is constant in body components,

$$\boxed{\mathbf{l}_b \dot{\boldsymbol{\omega}}_b + \boldsymbol{\omega}_b \times (\mathbf{l}_b \boldsymbol{\omega}_b) = \boldsymbol{\tau}_b.}$$

Call this Euler's rigid-body equation in vector form.

CHECK If $\mathbf{l}_b = II_3$ is isotropic, then $\boldsymbol{\omega} \times (\mathbf{l}\boldsymbol{\omega}) = I\boldsymbol{\omega} \times \boldsymbol{\omega} = 0$, so the equation reduces to $I\dot{\boldsymbol{\omega}} = \boldsymbol{\tau}$.

12–25 min: Principal axes and Euler’s component equations

BOARD Because $\mathbf{I}_b$ is real and symmetric, choose an orthonormal eigenbasis:

$$\mathbf{I}_b = \text{diag}(I_1, I_2, I_3).$$

Then

$$\mathbf{J}_b = \begin{pmatrix} I_1 \omega_1 \\ I_2 \omega_2 \\ I_3 \omega_3 \end{pmatrix}.$$

BOARD Expanding the vector equation gives

$$I_1 \dot{\omega}_1 + (I_3 - I_2) \omega_2 \omega_3 = \tau_1,$$

$$I_2 \dot{\omega}_2 + (I_1 - I_3) \omega_3 \omega_1 = \tau_2,$$

$$I_3 \dot{\omega}_3 + (I_2 - I_1) \omega_1 \omega_2 = \tau_3.$$

SAY This is another use of the Lecture 12 symmetric-matrix logic: choose the orthogonal eigenbasis of a quadratic form. But diagonal inertia does *not* diagonalize the nonlinear dynamics; the angular-velocity components remain coupled.

CHECK A principal-axis rotation $\boldsymbol{\omega} = \Omega \mathbf{e}_k$ makes the quadratic coupling vanish and is therefore an exact torque-free solution.

25–40 min: Torque-free motion: conserved quantities before solutions

BOARD Set

$$\boldsymbol{\tau} = \mathbf{0}.$$

Then in the laboratory frame

$$\mathbf{J}_s = \text{constant}.$$

Hence

$$J^2 = I_1^2 \omega_1^2 + I_2^2 \omega_2^2 + I_3^2 \omega_3^2 = \text{constant}.$$

BOARD Rotational kinetic energy is also conserved:

$$E = T_{\text{rot}} = \frac{1}{2} (I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2).$$

CHECK Verify energy conservation directly from Euler’s vector equation:

$$\frac{dT}{dt} = \boldsymbol{\omega} \cdot \dot{\boldsymbol{\omega}} = -\boldsymbol{\omega} \cdot [\boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega})] = 0.$$

SAY In body- $\boldsymbol{\omega}$ space, torque-free motion is confined to the intersection of the constant-energy ellipsoid and the constant- J^2 ellipsoid. We will use this geometry qualitatively; a full Poincaré construction is not required.

40–55 min: Steady principal-axis rotation and the intermediate-axis instability

BOARD Assume

$$I_1 < I_2 < I_3.$$

Every principal-axis rotation $\boldsymbol{\omega} = \Omega \mathbf{e}_k$ is an exact torque-free solution. Ask whether a small transverse perturbation remains small.

BOARD About axis 1, write

$$\omega_1 \simeq \Omega, \quad \omega_2 = \eta_2, \quad \omega_3 = \eta_3.$$

Linearized Euler equations give

$$\dot{\eta}_2 = \frac{I_3 - I_1}{I_2} \Omega \eta_3, \quad \dot{\eta}_3 = -\frac{I_2 - I_1}{I_3} \Omega \eta_2,$$

so

$$\ddot{\eta}_2 = -\Omega^2 \frac{(I_3 - I_1)(I_2 - I_1)}{I_2 I_3} \eta_2.$$

Axis 1 is linearly stable.

BOARD About axis 2,

$$\ddot{\eta}_1 = +\Omega^2 \frac{(I_3 - I_2)(I_2 - I_1)}{I_1 I_3} \eta_1,$$

so perturbations grow exponentially. Axis 2 is unstable.

BOARD About axis 3 the corresponding coefficient is negative, so axis 3 is stable.

smallest and largest principal moments: stable intermediate moment: unstable.

SAY This is the Lecture 12 stability move again: linearize about an exact solution, then let the sign of the perturbation equation decide local behavior.

CHECK Demonstrate with a book or rectangular object and identify the three principal-moment axes before tossing it.

55–68 min: Torque-free symmetric top: exact reduction by symmetry

BOARD Let

$$I_1 = I_2 = I_\perp, \quad I_3 = I_\parallel.$$

Euler's equations with zero torque give

$$\dot{\omega}_3 = 0$$

and

$$\dot{\omega}_1 = -\Omega_b \omega_2, \quad \dot{\omega}_2 = \Omega_b \omega_1,$$

where

$$\Omega_b = \frac{I_\parallel - I_\perp}{I_\perp} \omega_3.$$

Thus the transverse body components rotate uniformly:

$$\omega_1 + i\omega_2 = C e^{i\Omega_b t}.$$

SAY In body coordinates the angular-velocity components precess around the symmetry axis. In space, the torque-free angular momentum vector $\mathbf{J}_s$ is fixed.

BOARD Let

$$\mathbf{n} = Q \mathbf{e}_3$$

be the body symmetry axis in laboratory components. Kinematics gives

$$\dot{\mathbf{n}} = \boldsymbol{\omega}_s \times \mathbf{n}.$$

For a symmetric body,

$$\mathbf{J}_s = I_\perp \boldsymbol{\omega}_s + (I_\parallel - I_\perp)(\boldsymbol{\omega}_s \cdot \mathbf{n})\mathbf{n}.$$

Cross with $\mathbf{n}$:

$$\mathbf{J}_s \times \mathbf{n} = I_\perp \boldsymbol{\omega}_s \times \mathbf{n} = I_\perp \dot{\mathbf{n}}.$$

Therefore

$$\dot{\mathbf{n}} = \frac{1}{I_\perp} \mathbf{J}_s \times \mathbf{n}.$$

The symmetry axis precesses uniformly about the fixed $\mathbf{J}_s$ with space-frame rate $J/I_\perp$.

STOP Exam II rigid-body boundary. Required rigid-body material ends here: Lecture 13 kinematics plus Euler equations, principal axes, torque-free invariants, intermediate-axis stability, and the torque-free symmetric top. PSet6 Q15–Q16 provide the corresponding study problems. The remainder of this lecture is conceptual extension and is not part of Exam II.

68–78 min: Orientation representations: choose a description for a purpose

BOARD The orientation itself is the map

$$Q \in SO(3).$$

Three common descriptions emphasize different structures.

Description	Advantage	Tradeoff
Q	Direct frame map; acts immediately on points and axes; globally nonsingular	Nine entries with orthogonality constraints
Euler coordinates	Three local scalar generalized coordinates; cyclic coordinates can make reduction transparent	Convention dependent; coordinate singularities; angle rates are not angular-velocity components
Rotor $\mathcal{R}$	Compact geometric composition; no coordinate singularity; natural generator/bivector relation	Unit constraint, double-cover sign, and less familiar notation

SAY Euler angles are not a different physical orientation object. They are one local coordinate choice or factorization of the same Q . Their value is greatest when scalar generalized coordinates expose symmetries or cyclic variables.

OPTIONAL Rotor reminder only:

$$\mathbf{a}_s = \mathcal{R} \mathbf{a}_b \tilde{\mathcal{R}}, \quad \mathcal{R} \tilde{\mathcal{R}} = 1, \quad \mathcal{R} \sim -\mathcal{R}.$$

Do not repeat the Lecture 3/13 GA development here.

78–88 min: Symmetry suggests the decomposition of orientation

BOARD For a symmetric top, the distinguished physical direction is

$$\mathbf{n} = Q \mathbf{e}_3.$$

Instead of choosing three generic angles immediately, factor the orientation as

$$Q = Q_{\mathbf{n}} R_3(\chi),$$

where

$$Q_{\mathbf{n}} \mathbf{e}_3 = \mathbf{n}, \quad R_3(\chi) \mathbf{e}_3 = \mathbf{e}_3.$$

SAY The decomposition separates the direction of the symmetry axis from rotation about that axis:

$$\text{orientation} = \text{axis direction} + \text{spin about the axis.}$$

BOARD The split is not unique:

$$Q_{\mathbf{n}} \rightarrow Q_{\mathbf{n}} R_3(\alpha), \quad \chi \rightarrow \chi - \alpha.$$

Only the product Q is physical. An Euler convention is one particular local choice of $Q_{\mathbf{n}}$ together with the residual angle χ .

CHECK For example, one may choose locally

$$Q_{\mathbf{n}} = R_z(\phi) R_y(\theta),$$

so that

$$Q = R_z(\phi) R_y(\theta) R_3(\chi).$$

This is an Euler-type decomposition, obtained *after* identifying the symmetry-adapted variables.

88–100 min: Heavy symmetric top: reduce to the symmetry axis and two constants

BOARD Fix the pivot. Let the center of mass lie a distance ℓ along the body symmetry axis. Then

$$\mathbf{n} = Q\mathbf{e}_3, \quad \mathbf{r}_{\text{CM}} = \ell\mathbf{n}.$$

Take $\mathbf{e}_z$ upward, so

$$V = Mg\ell \mathbf{e}_z \cdot \mathbf{n}.$$

Use principal moments about the pivot,

$$\mathbf{I}_O = \text{diag}(I_\perp, I_\perp, I_\parallel).$$

OPTIONAL If inertia about the center of mass is known, the pivot tensor follows from the parallel-axis theorem:

$$\mathbf{I}_O = \mathbf{I}_{\text{CM}} + M\ell^2(I_3 - \mathbf{e}_3\mathbf{e}_3^T).$$

BOARD Since

$$\dot{\mathbf{n}} = \boldsymbol{\omega} \times \mathbf{n},$$

the angular velocity can be decomposed without Euler angles:

$$\boldsymbol{\omega} = \mathbf{n} \times \dot{\mathbf{n}} + \sigma\mathbf{n}, \quad \sigma = \boldsymbol{\omega} \cdot \mathbf{n}.$$

Therefore

$$T = \frac{1}{2}I_\perp |\dot{\mathbf{n}}|^2 + \frac{1}{2}I_\parallel \sigma^2.$$

BOARD Rotation about the body symmetry axis gives the conserved quantity

$$J_3 = I_\parallel \sigma = \text{constant}.$$

Vertical rotational symmetry gives

$$J_z = \mathbf{e}_z \cdot \mathbf{J} = \text{constant}.$$

Moreover

$$\mathbf{J} = I_\perp(\mathbf{n} \times \dot{\mathbf{n}}) + J_3\mathbf{n}.$$

BOARD Now introduce coordinates only on the sphere of possible axis directions:

$$\mathbf{n} = (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta).$$

Then

$$|\dot{\mathbf{n}}|^2 = \dot{\theta}^2 + \sin^2\theta \dot{\phi}^2,$$

and

$$(\mathbf{n} \times \dot{\mathbf{n}}) \cdot \mathbf{e}_z = \sin^2\theta \dot{\phi}.$$

Thus

$$J_z = I_\perp \sin^2\theta \dot{\phi} + J_3 \cos\theta,$$

so

$$\dot{\phi} = \frac{J_z - J_3 \cos\theta}{I_\perp \sin^2\theta}.$$

BOARD Energy is

$$E = \frac{1}{2}I_\perp \left(\dot{\theta}^2 + \sin^2\theta \dot{\phi}^2 \right) + \frac{J_3^2}{2I_\parallel} + Mg\ell \cos\theta.$$

Eliminate $\dot{\phi}$:

$$\frac{1}{2}I_\perp \dot{\theta}^2 + V_{\text{eff}}(\theta) = E - \frac{J_3^2}{2I_\parallel},$$

with

$$V_{\text{eff}}(\theta) = Mgl \cos \theta + \frac{(J_z - J_3 \cos \theta)^2}{2I_{\perp} \sin^2 \theta}.$$

SAY This is the standard heavy-top nutation quadrature, obtained without parameterizing the full orientation by three Euler angles. Only the direction $\mathbf{n} \in S^2$ needed the two coordinates (θ, ϕ) ; the residual spin was already represented by the conserved J_3 .

100–105 min: Euler angles reappear as one reconstruction choice

BOARD If we choose the local section

$$Q_{\mathbf{n}} = R_z(\phi)R_y(\theta)$$

and write

$$Q = Q_{\mathbf{n}}R_3(\chi),$$

then the physical spin rate along the symmetry axis is

$$\sigma = \boldsymbol{\omega} \cdot \mathbf{n} = \dot{\chi} + \dot{\phi} \cos \theta.$$

SAY The familiar Euler-angle combination is now geometric rather than mnemonic: the spin about $\mathbf{n}$ equals the explicit residual rotation rate plus the rotation about $\mathbf{n}$ induced by moving the chosen frame $Q_{\mathbf{n}}$ over the sphere.

BOARD Final representation map:

$$\begin{array}{l} Q : \text{full orientation and direct frame map} \\ Q = Q_{\mathbf{n}}R_3(\chi) : \begin{array}{l} \Downarrow \text{use symmetry} \\ \text{axis direction} + \text{residual spin} \end{array} \\ \Downarrow \text{choose local coordinates only when useful} \\ \mathbf{n} \leftrightarrow (\theta, \phi), \quad \chi \text{ reconstructs the full orientation} \end{array}$$

NEXT After Exam II, Hamiltonian mechanics will systematize ideas already visible here: cyclic variables, conserved momenta, reduction, and effective one-dimensional dynamics.

Pacing and stopping rule. Protect the vector Euler equation, principal-axis equations, torque-free invariants, intermediate-axis stability, and torque-free symmetric top before the Exam II boundary. After the boundary, protect the conceptual statement that Euler angles are one decomposition choice for Q , the symmetry-adapted factorization $Q = Q_{\mathbf{n}}R_3(\chi)$, and the heavy-top effective potential. If time is short, shorten the conserved-ellipsoid discussion, the representation table, the gauge-freedom comment, and the parallel-axis aside before cutting those structural steps.