Lecture 13 Instructor Outline — Revised Draft

Rigid Bodies I: Moving Frames, Angular Velocity, and Inertia

PHYS V1100: Analytical Dynamics — 105 minutes

Point matrices, distance geometry, rotation matrices, and rigid-body kinematics

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Core arc:

$$
\begin{array}{c}
X \xrightarrow{\text{affine frame}} Y = A\mathbf{1}^T + BX \xrightarrow{\text{preserve } \mathsf{D}^2} B^TB = I \\
\Downarrow \\
X(t) = \boldsymbol{R}(t)\mathbf{1}^T + Q(t)X_0, \qquad X_0\mathsf{M}\mathbf{1} = 0 \\
\Downarrow \\
Q = [\boldsymbol{e}_1^{(b)}\ \boldsymbol{e}_2^{(b)}\ \boldsymbol{e}_3^{(b)}] \xrightarrow{\text{how does the frame move?}} \dot{Q} = W_sQ \\
\Downarrow \\
W_s^T = -W_s, \qquad W_s\boldsymbol{r} = \boldsymbol{\omega}_s \times \boldsymbol{r} \\
\Downarrow \\
\dot{\boldsymbol{x}}_i = \dot{\boldsymbol{R}} + \boldsymbol{\omega}_s \times (\boldsymbol{x}_i - \boldsymbol{R}) \to T_{\text{rot}} = \tfrac{1}{2}\boldsymbol{\omega}_b^T\mathsf{I}_b\boldsymbol{\omega}_b \to \boldsymbol{J}_b = \mathsf{I}_b\boldsymbol{\omega}_b
\end{array}
$$

Protected goals.

- Treat a rigid body first as a collection of material points arranged as a $d \times N$ matrix.
- Let preservation of the pairwise-distance matrix motivate $Q^TQ = I$ rather than assuming a rotation matrix at the outset.
- Make the reference configuration X_0 and moving frame $[\boldsymbol{R}\ Q]$ the primary description of rigid configuration.
- Derive the center of mass as the body-origin choice $X_0\mathsf{M}\mathbf{1} = 0$ and show why this removes the translation–rotation kinetic cross term.
- Motivate W_s as the instantaneous linear map that moves the body frame; then let angular velocity emerge from skew symmetry in three dimensions.
- Start from the original particle kinetic energy and let the second-moment matrix and inertia tensor emerge.
- Stop before Euler equations, tops, or a required Euler-angle parameterization.

Notation and scope. $X(t) = [\boldsymbol{x}_1(t)\ \cdots\ \boldsymbol{x}_N(t)] \in \mathbb{R}^{d\times N}$ is the current point matrix and $X_0 = X(0)$ is a chosen reference configuration. This is distinct from the stacked dN-vector used earlier with incidence/compatibility operators. $\mathbf{1} \in \mathbb{R}^N$ is the all-ones vector, $\boldsymbol{R} \in \mathbb{R}^d$ is the body-origin position, $Q \in SO(d)$ is the moving orthonormal frame, and $\mathsf{M} = \operatorname{diag}(m_1, \ldots, m_N)$ is the particle mass matrix. In the three-dimensional kinematic sections, subscripts s, b denote space/laboratory and body components of the same physical quantity.

Pacing. The first 42 minutes build rigid motion from distance geometry and choose a center-of-mass reference frame; the next 28 derive instantaneous frame motion and the rigid velocity field; the final 35 obtain inertia and angular momentum from the particle description.

0–18 min: Point matrices, Gram matrices, and what rigidity preserves

BOARD Arrange the N material points as columns:

$$\boxed{X = [\boldsymbol{x}_1\ \boldsymbol{x}_2\ \cdots\ \boldsymbol{x}_N] \in \mathbb{R}^{d\times N}.}$$

SAY This is the same point data we have assembled before, but in a different shape. The earlier stacked vector

$$\boldsymbol{x}_{\text{stack}} = \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \vdots \\ \boldsymbol{x}_N \end{bmatrix} \in \mathbb{R}^{dN}$$

is natural for compatibility maps such as $(A \otimes I_d)\boldsymbol{x}_{\text{stack}}$. The $d \times N$ matrix X is natural when the *same physical-space map* acts on every point from the left.

BOARD Define the Gram matrix

$$\boxed{\mathsf{G}(X) = X^T X, \qquad G_{ij} = \boldsymbol{x}_i^T \boldsymbol{x}_j.}$$

Its diagonal gives squared lengths; its off-diagonal entries give mutual inner products. This is the same "Gram" that appears in Gram–Schmidt orthogonalization.

BOARD Let

$$\boldsymbol{g} = \operatorname{diag}(X^T X)$$

denote the column vector of diagonal entries. Since

$$d_{ij}^2 = (\boldsymbol{x}_j - \boldsymbol{x}_i)^T (\boldsymbol{x}_j - \boldsymbol{x}_i) = G_{ii} + G_{jj} - 2G_{ij},$$

the complete squared-distance matrix is

$$\boxed{\mathsf{D}^2(X) = \boldsymbol{g}\mathbf{1}^T + \mathbf{1}\boldsymbol{g}^T - 2X^T X.}$$

ASK Which of these objects should define the shape of a free rigid body: $X^T X$ or $\mathsf{D}^2(X)$?

CHECK $X^T X$ depends on the chosen origin. $\mathsf{D}^2(X)$ does not: it stores only internal pairwise distances.

BOARD Now apply a general affine frame map

$$\boxed{Y = A\mathbf{1}^T + BX,} \qquad A \in \mathbb{R}^d, \quad B \in GL(d).$$

Column by column,

$$\boldsymbol{y}_i = A + B\boldsymbol{x}_i.$$

Therefore

$$\boxed{\boldsymbol{y}_j - \boldsymbol{y}_i = B(\boldsymbol{x}_j - \boldsymbol{x}_i).}$$

The translation A drops out before any calculation of distance.

BOARD Hence

$$\boxed{d_{ij}^2(Y) = (\boldsymbol{x}_j - \boldsymbol{x}_i)^T B^T B (\boldsymbol{x}_j - \boldsymbol{x}_i).}$$

Equivalently, the Gram matrix entering the distance formula is replaced by

$$X^T X \quad \longrightarrow \quad X^T B^T B X.$$

Thus

$$\mathsf{D}^2(Y) = \boldsymbol{g}_B \mathbf{1}^T + \mathbf{1}\boldsymbol{g}_B^T - 2X^T B^T B X, \qquad \boldsymbol{g}_B = \operatorname{diag}(X^T B^T B X).$$

ASK What restriction on B makes every internal distance unchanged?

BOARD Certainly

$$\boxed{B^T B = I_d \quad \Longrightarrow \quad \mathsf{D}^2(Y) = \mathsf{D}^2(X).}$$

For a body whose relative displacement vectors span $\mathbb{R}^d$, the converse also holds. Choose d independent relative vectors $\boldsymbol{v}_a$. Preserving $|\boldsymbol{v}_a|$, $|\boldsymbol{v}_b|$, and $|\boldsymbol{v}_a - \boldsymbol{v}_b|$ gives, by polarization,

$$(B\boldsymbol{v}_a)^T (B\boldsymbol{v}_b) = \boldsymbol{v}_a^T \boldsymbol{v}_b.$$

Therefore B preserves inner products on a basis and

$$\boxed{B^T B = I_d.}$$

SAY Rigidity has forced the linear part of the affine map to be orthogonal. Then $B^{-1} = B^T$ and $\det B = \pm 1$. Ordinary rigid motion connected continuously to the identity uses the orientation-preserving branch

$$\boxed{Q^T Q = QQ^T = I_d, \qquad Q^{-1} = Q^T, \qquad \det Q = +1.}$$

The name $SO(d)$ is optional; the geometry is not.

STOP If the point set lies in a lower-dimensional affine subspace, its distance data constrains B only on the span of its relative displacement vectors. For an ordinary three-dimensional body, assume enough noncoplanar material points to determine the full orientation.

18–30 min: Choose a reference body and follow one moving frame

BOARD Having identified the rigid affine map, choose one configuration as the reference body:

$$\boxed{X_0 = X(0).}$$

Choose the reference spatial frame so that

$$\boxed{\boldsymbol{R}(0) = \mathbf{0}, \qquad Q(0) = I_d.}$$

Then every rigid configuration is

$$\boxed{X(t) = \boldsymbol{R}(t)\mathbf{1}^T + Q(t)X_0.}$$

Column by column,

$$\boxed{\boldsymbol{x}_i(t) = \boldsymbol{R}(t) + Q(t)\boldsymbol{x}_{i0}.}$$

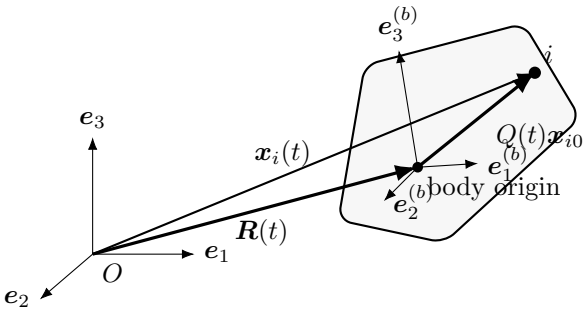
SAY The reference coordinates $\boldsymbol{x}_{i0}$ are fixed. All dynamics of the ideal rigid body are carried by the moving origin $\boldsymbol{R}(t)$ and the moving frame $Q(t)$.

BOARD In three dimensions the laboratory basis is

$$[\boldsymbol{e}_1\ \boldsymbol{e}_2\ \boldsymbol{e}_3] = I_3,$$

while the body-attached axes expressed in laboratory components are the columns of

$$\boxed{Q(t) = [\boldsymbol{e}_1^{(b)}(t)\ \boldsymbol{e}_2^{(b)}(t)\ \boldsymbol{e}_3^{(b)}(t)].}$$



BOARD Compress the same statement into one block product:

$$\boxed{X(t) = [\boldsymbol{R}(t)\ Q(t)] \begin{bmatrix} \mathbf{1}^T \\ X_0 \end{bmatrix}.}$$

SAY Follow the frame; rigidity makes all the material points follow.

CHECK Directly,

$$\boldsymbol{x}_j(t) - \boldsymbol{x}_i(t) = Q(t)(\boldsymbol{x}_{j0} - \boldsymbol{x}_{i0}),$$

so $Q^TQ = I$ reproduces the fixed-distance condition from the opening.

30–42 min: Why choose the center of mass as the body origin?

BOARD Let the particle mass matrix be

$$\boxed{\mathsf{M} = \operatorname{diag}(m_1, \ldots, m_N),} \qquad \boxed{m_T = \mathbf{1}^T\mathsf{M}\mathbf{1}.}$$

Suppose the current reference coordinates have an arbitrary body origin. Their mass-weighted mean is the *vector*

$$\boxed{\boldsymbol{c} = \frac{X_0\mathsf{M}\mathbf{1}}{m_T} \in \mathbb{R}^d.}$$

BOARD Recenter every reference point by the same vector:

$$\boxed{X_{0c} = X_0 - \boldsymbol{c}\mathbf{1}^T.}$$

Then

$$X_{0c}\mathsf{M}\mathbf{1} = X_0\mathsf{M}\mathbf{1} - \boldsymbol{c}\,\mathbf{1}^T\mathsf{M}\mathbf{1} = \mathbf{0}.$$

Thus the center-of-mass body frame is characterized by

$$\boxed{X_0\mathsf{M}\mathbf{1} = \mathbf{0},}$$

after dropping the temporary subscript c.

BOARD The same physical configuration is unchanged because

$$\boldsymbol{R}\mathbf{1}^T + Q(X_0 + \boldsymbol{c}\mathbf{1}^T) = (\boldsymbol{R} + Q\boldsymbol{c})\mathbf{1}^T + QX_0.$$

Changing the body origin shifts $\boldsymbol{R}$ by $Q\boldsymbol{c}$ and subtracts $\boldsymbol{c}$ from every body coordinate.

BOARD Now multiply the rigid-body map by $\mathsf{M}\mathbf{1}$:

$$X(t)\mathsf{M}\mathbf{1} = \boldsymbol{R}\,\mathbf{1}^T\mathsf{M}\mathbf{1} + QX_0\mathsf{M}\mathbf{1} = m_T\boldsymbol{R}.$$

Therefore

$$\boxed{\boldsymbol{R}(t) = \frac{X(t)\mathsf{M}\mathbf{1}}{\mathbf{1}^T\mathsf{M}\mathbf{1}}.}$$

SAY The usual center-of-mass formula is exactly the translation coordinate obtained by choosing the body origin so that the fixed reference mass distribution has zero first moment.

SAY Preview the second payoff: the same condition $X_0\mathsf{M}\mathbf{1} = 0$ will remove the translation–rotation cross term from kinetic energy.

42–58 min: How is the moving frame changing?

BOARD Return to

$$Q = [\boldsymbol{e}_1^{(b)}\ \boldsymbol{e}_2^{(b)}\ \boldsymbol{e}_3^{(b)}].$$

Then

$$\dot{Q} = [\dot{\boldsymbol{e}}_1^{(b)}\ \dot{\boldsymbol{e}}_2^{(b)}\ \dot{\boldsymbol{e}}_3^{(b)}].$$

ASK Is there one linear map that takes every *current* body axis into its instantaneous velocity?

BOARD Because the current axes form a basis, there is a unique map W_s such that

$$\boxed{\dot{\boldsymbol{e}}_a^{(b)} = W_s\boldsymbol{e}_a^{(b)}, \qquad a = 1, 2, 3.}$$

Collect the three columns:

$$\boxed{\dot{Q} = W_sQ.}$$

Since $Q^{-1} = Q^T$,

$$\boxed{W_s = \dot{Q}Q^T.}$$

SAY W_s is not introduced by a matrix trick. It is the instantaneous linear map that moves the body frame, expressed in laboratory components.

BOARD Rigidity now determines the kind of map W_s can be. Differentiate

$$QQ^T = I_3 :$$

$$\dot{Q}Q^T + Q\dot{Q}^T = 0.$$

Therefore

$$\boxed{W_s^T = -W_s.}$$

BOARD Every real 3×3 skew matrix has the form

$$W_s = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}, \qquad \boxed{W_s \boldsymbol{r} = \boldsymbol{\omega}_s \times \boldsymbol{r}.}$$

SAY Angular velocity has now emerged as the three-component vector representation of the instantaneous skew map that moves an orthonormal frame. Finite rotations are not additive vectors; their instantaneous generators form a three-dimensional linear space.

CHECK If $\boldsymbol{\omega}_s = \omega \boldsymbol{e}_3$, then $W_s \boldsymbol{e}_1 = \omega \boldsymbol{e}_2$ and $W_s \boldsymbol{e}_2 = -\omega \boldsymbol{e}_1$. Predict the motion of the frame before multiplying matrices.

OPTIONAL One-line GA callback only: W_s may equally be represented by an angular-velocity bivector, an oriented plane with magnitude $|\boldsymbol{\omega}|$. Do not develop rotors or the half-angle construction in the protected lecture.

58–70 min: The velocity of every body point follows from the frame

BOARD Define the displacement of material point i from the COM:

$$\boxed{\boldsymbol{r}_i(t) = \boldsymbol{x}_i(t) - \boldsymbol{R}(t) = Q(t) \boldsymbol{x}_{i0}.}$$

Differentiate:

$$\dot{\boldsymbol{r}}_i = \dot{Q} \boldsymbol{x}_{i0} = W_s Q \boldsymbol{x}_{i0} = W_s \boldsymbol{r}_i.$$

Hence

$$\boxed{\dot{\boldsymbol{r}}_i = \boldsymbol{\omega}_s \times \boldsymbol{r}_i,}$$

and therefore

$$\boxed{\dot{\boldsymbol{x}}_i = \dot{\boldsymbol{R}} + \boldsymbol{\omega}_s \times (\boldsymbol{x}_i - \boldsymbol{R}).}$$

One translational velocity and one angular velocity determine the velocity of every point of the rigid body.

BOARD For the whole point matrix,

$$\boxed{\dot{X} = \dot{\boldsymbol{R}} \mathbf{1}^T + W_s \big(X - \boldsymbol{R} \mathbf{1}^T\big).}$$

CHECK $\boldsymbol{r}_i^T \dot{\boldsymbol{r}}_i = \boldsymbol{r}_i^T W_s \boldsymbol{r}_i = 0$ because W_s is skew. Thus each point's instantaneous relative velocity is perpendicular to its radius from the body origin, exactly as fixed distance requires.

BOARD Describe the same instantaneous rotation in body components:

$$\boxed{W_b = Q^T \dot{Q}, \qquad W_b^T = -W_b.}$$

Then

$$\boxed{\dot{Q} = W_s Q = Q W_b,} \qquad \boxed{W_b = Q^T W_s Q.}$$

In three dimensions,

$$\boxed{\boldsymbol{\omega}_b = Q^T \boldsymbol{\omega}_s.}$$

BOARD If a physical vector is represented by $\boldsymbol{b}_s = Q \boldsymbol{b}_b$, then

$$\dot{\boldsymbol{b}}_s = \dot{Q} \boldsymbol{b}_b + Q \dot{\boldsymbol{b}}_b = W_s \boldsymbol{b}_s + Q \dot{\boldsymbol{b}}_b,$$

so

$$\boxed{Q^T \dot{\boldsymbol{b}}_s = \dot{\boldsymbol{b}}_b + \boldsymbol{\omega}_b \times \boldsymbol{b}_b.}$$

SAY This is the product rule for a moving basis, not a new force law. Laboratory axes simplify inertial differentiation; body axes will make the mass distribution and inertia tensor constant.

BOARD Configuration versus velocity-level state:

$$\boxed{\text{configuration: } (\boldsymbol{R}, Q), \qquad \text{convenient state data: } (\boldsymbol{R}, Q, \dot{\boldsymbol{R}}, \boldsymbol{\omega}).}$$

70–90 min: Let inertia emerge from the particle kinetic energy

BOARD Start from the particle kinetic energy, written in point-matrix form:

$$\boxed{T = \frac{1}{2}\,\mathrm{tr}(\dot{X}\mathsf{M}\dot{X}^T).}$$

This is exactly $\frac{1}{2}\sum_i m_i \dot{\boldsymbol{x}}_i^T \dot{\boldsymbol{x}}_i$.

BOARD From

$$X = \boldsymbol{R}\mathbf{1}^T + QX_0$$

we have

$$\dot{X} = \dot{\boldsymbol{R}}\mathbf{1}^T + \dot{Q}X_0.$$

Substitute and expand only enough to expose the three pieces:

$$T = \frac{1}{2}m_T\dot{\boldsymbol{R}}^T\dot{\boldsymbol{R}} + \dot{\boldsymbol{R}}^T\dot{Q}X_0\mathsf{M}\mathbf{1} + \frac{1}{2}\,\mathrm{tr}(\dot{Q}X_0\mathsf{M}X_0^T\dot{Q}^T).$$

CHECK The mixed term vanishes precisely because the body origin is the COM:

$$\boxed{X_0\mathsf{M}\mathbf{1} = 0.}$$

Thus

$$\boxed{T = \frac{1}{2}m_T|\dot{\boldsymbol{R}}|^2 + T_{\mathrm{rot}},} \qquad \boxed{T_{\mathrm{rot}} = \frac{1}{2}\,\mathrm{tr}(\dot{Q}X_0\mathsf{M}X_0^T\dot{Q}^T).}$$

SAY This is the promised dynamical reason for the center-of-mass definition: it is the body-origin choice that removes the translation–rotation kinetic cross term. The split is exact for kinetic energy; an external potential need not separate in the same way.

BOARD Define the constant body-frame second-moment matrix

$$\boxed{\mathsf{S}_b = X_0\mathsf{M}X_0^T.}$$

Then, using $\dot{Q} = QW_b$ and cyclic invariance of the trace,

$$\boxed{T_{\mathrm{rot}} = \frac{1}{2}\,\mathrm{tr}(W_b\mathsf{S}_bW_b^T).}$$

BOARD Return briefly to the particle columns. Since $W_b\boldsymbol{x}_{i0} = \boldsymbol{\omega}_b \times \boldsymbol{x}_{i0}$,

$$T_{\mathrm{rot}} = \frac{1}{2}\sum_i m_i|\boldsymbol{\omega}_b \times \boldsymbol{x}_{i0}|^2.$$

Use

$$|\boldsymbol{\omega} \times \boldsymbol{x}|^2 = \boldsymbol{\omega}^T\left(|\boldsymbol{x}|^2 I_3 - \boldsymbol{x}\boldsymbol{x}^T\right)\boldsymbol{\omega}.$$

Therefore

$$\boxed{T_{\mathrm{rot}} = \frac{1}{2}\boldsymbol{\omega}_b^T\mathsf{I}_b\boldsymbol{\omega}_b,}$$

with

$$\boxed{\mathsf{I}_b = \sum_i m_i\left(|\boldsymbol{x}_{i0}|^2 I_3 - \boldsymbol{x}_{i0}\boldsymbol{x}_{i0}^T\right).}$$

Because

$$\mathrm{tr}(\mathsf{S}_b) = \sum_i m_i|\boldsymbol{x}_{i0}|^2,$$

the same result is

$$\boxed{\mathsf{I}_b = \mathrm{tr}(\mathsf{S}_b)I_3 - \mathsf{S}_b.}$$

SAY The matrix $\mathsf{S}_b = X_0 \mathsf{M} X_0^T$ is the mass-weighted second moment of the reference point cloud. The familiar inertia tensor is obtained from it by subtracting the directional second moment from the total second moment.

CHECK For rotation about a unit axis $\hat{\boldsymbol{n}}$,

$$I_{\hat{\boldsymbol{n}}} = \hat{\boldsymbol{n}}^T \mathsf{I}_b \hat{\boldsymbol{n}} = \sum_i m_i r_{i,\perp}^2.$$

Distance perpendicular to the axis, not distance along the axis, sets rotational inertia.

90–100 min: Angular momentum and why body coordinates help

BOARD Angular momentum about the center of mass is

$$\boldsymbol{J}_s = \sum_i \boldsymbol{r}_i \times m_i \dot{\boldsymbol{r}}_i.$$

With $\dot{\boldsymbol{r}}_i = \boldsymbol{\omega}_s \times \boldsymbol{r}_i$,

$$\boldsymbol{r}_i \times (\boldsymbol{\omega}_s \times \boldsymbol{r}_i) = |\boldsymbol{r}_i|^2 \boldsymbol{\omega}_s - \boldsymbol{r}_i (\boldsymbol{r}_i \cdot \boldsymbol{\omega}_s),$$

so

$$\boxed{\boldsymbol{J}_s = \mathsf{I}_s \boldsymbol{\omega}_s,} \qquad \boxed{T_{\text{rot}} = \frac{1}{2} \boldsymbol{\omega}_s \cdot \boldsymbol{J}_s.}$$

BOARD The body mass distribution is fixed, so

$$\boxed{\mathsf{I}_s(t) = Q \mathsf{I}_b Q^T,}$$

and therefore

$$\boxed{\boldsymbol{J}_b = Q^T \boldsymbol{J}_s = \mathsf{I}_b \boldsymbol{\omega}_b.}$$

SAY This is the fundamental tradeoff:

space frame: inertial differentiation is simple,
body frame: X_0, S_b, I_b are constant.

The moving-frame derivative lets us use both descriptions without confusing them.

ASK Must $\boldsymbol{J}$ point along $\boldsymbol{\omega}$?

CHECK Only when $\boldsymbol{\omega}$ is an eigenvector of I (a principal-axis direction), or when the relevant inertia eigenvalues are degenerate. Lecture 12's symmetric-matrix reasoning will return here.

OPTIONAL If time permits, write in principal axes

$$\mathsf{I}_b = \operatorname{diag}(I_1, I_2, I_3),$$

but do not yet derive Euler's equations.

100–105 min: One-board summary and the next question

BOARD Keep the structural chain together:

$$\boxed{X(t) = \boldsymbol{R}(t) \mathbf{1}^T + Q(t) X_0,} \qquad \boxed{X_0 \mathsf{M} \mathbf{1} = 0,}$$

$$\boxed{\boldsymbol{R} = \frac{X \mathsf{M} \mathbf{1}}{\mathbf{1}^T \mathsf{M} \mathbf{1},}} \qquad \boxed{\dot{Q} = W_s Q = Q W_b,}$$

$$\boxed{W_s^T = -W_s, \qquad W_s \boldsymbol{r} = \boldsymbol{\omega}_s \times \boldsymbol{r},}$$

$$\boxed{\dot{\boldsymbol{x}}_i = \dot{\boldsymbol{R}} + \boldsymbol{\omega}_s \times (\boldsymbol{x}_i - \boldsymbol{R}),}$$

$$\boxed{T = \frac{1}{2} m_T |\dot{\boldsymbol{R}}|^2 + \frac{1}{2} \boldsymbol{\omega}_b^T \mathsf{I}_b \boldsymbol{\omega}_b,} \qquad \boxed{\boldsymbol{J}_b = \mathsf{I}_b \boldsymbol{\omega}_b.}$$

ASK If there is no external torque about the center of mass, laboratory angular momentum is constant. Does that imply angular velocity is constant?

NEXT Not in general. The next dynamical step is to combine

$$\dot{\boldsymbol{J}}_s = \boldsymbol{\tau}_s, \qquad Q^T \dot{\boldsymbol{J}}_s = \dot{\boldsymbol{J}}_b + \boldsymbol{\omega}_b \times \boldsymbol{J}_b, \qquad \boldsymbol{J}_b = \mathsf{I}_b \boldsymbol{\omega}_b,$$

which leads to Euler's equations and torque-free rigid-body motion.

Representations deliberately deferred from the protected core.

Euler angles. Introduce them when three local generalized coordinates for orientation are actually needed, most naturally in the top/precession discussion. Then $Q = Q(\boldsymbol{q}_{\text{rot}})$ and

$$\boldsymbol{\omega}_b = E(\boldsymbol{q}_{\text{rot}})\dot{\boldsymbol{q}}_{\text{rot}}$$

has a clear purpose. Do not suggest that $\boldsymbol{\omega} = (\dot{\phi}, \dot{\theta}, \dot{\psi})$.

Geometric algebra. Keep the Lecture 3 bivector viewpoint available as a parallel representation of instantaneous rotation. The rotor $\mathcal{R} = e^{-B\theta/2}$, half-angle check, rotor composition, and rotor differential equation can be developed in expanded notes or a dedicated callback if they become useful; they are not required for the matrix-based rigid-body derivation above.

Pacing and stopping rule. Protect the distance-matrix derivation of orthogonality, the reference-body map, the COM recentering and kinetic cross-term payoff, the frame-based motivation of W_s, the rigid velocity field, and the derivation of I_b from particle kinetic energy. If time is short, shorten the converse proof $\mathsf{D}^2 \Rightarrow B^T B = I$, the full space/body derivative discussion, and the principal-axis preview before cutting any of those protected pieces.