Lecture 12 Instructor Outline — Draft
Small Oscillations II and Beyond Linearization

105 minutes — normal coordinates, symmetry, stability, and nonlinear phase space

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Core arc:

$$K\boldsymbol{u}_\alpha = \omega_\alpha^2 M\boldsymbol{u}_\alpha$$
$$\Downarrow$$
mass weighting $\to$ M-orthogonality $\to$ normal coordinates
$$\Downarrow$$
independent quadratic modes $\to$ degeneracy, symmetry, and zero modes
$$\Longrightarrow$$
restore higher-order terms $\to$ mode coupling and amplitude dependence
$$\Downarrow$$
nonlinear phase portrait $\to$ limits of linearization

Protected goals:

- Begin from the generalized eigenproblem established in Lecture 11 rather than re-deriving the quadratic approximation.
- Show why the natural modal inner product is mass weighted: $\boldsymbol{u}_\alpha^T M\boldsymbol{u}_\beta$.
- Construct normal coordinates that simultaneously reduce $M \to I$ and $K \to \Omega^2$.
- Carry the component and matrix modal descriptions in parallel: $\mathbf{v}_\alpha \leftrightarrow V$, $\boldsymbol{u}_\alpha \leftrightarrow U$, and $a_\alpha \leftrightarrow \boldsymbol{a}$.
- Treat zero modes separately: $a_0(t) = a_0(0) + \dot{a}_0(0)t$.
- Distinguish degeneracy from a zero mode; use symmetry to explain why a degenerate eigenspace has no unique preferred basis.
- Connect finite-dimensional mass-weighted orthogonality to continuum mode orthogonality with density as the weight.
- Make explicit what the quadratic theory discarded: higher-order potential terms and, in general coordinates, higher-order kinetic couplings.
- Use the exact pendulum phase portrait to distinguish local harmonic motion, separatrix motion, and rotation.
- End with the stability hierarchy: positive quadratic form, negative direction, and marginal zero-curvature direction where quadratic theory is inconclusive.

0–10 min: Recover the generalized mode problem and mass-weight it

BOARD From Lecture 11:

$$\boxed{M\ddot{\boldsymbol{\xi}} + K\boldsymbol{\xi} = \mathbf{0},} \qquad \boxed{K\boldsymbol{u}_\alpha = \omega_\alpha^2 M\boldsymbol{u}_\alpha.}$$

Assume for the present discussion

$$\boxed{M = M^T > 0, \qquad K = K^T.}$$

SAY The generalized eigenproblem is not merely a harder version of an ordinary eigenproblem. The matrix M defines the kinetic-energy geometry of configuration space.

ASK Can we choose coordinates in which that kinetic-energy geometry becomes Euclidean?

BOARD Let $M^{1/2}$ be the positive-definite square root of M, and define the mass-weighted coordinate

$$\boxed{\boldsymbol{y} = M^{1/2}\boldsymbol{\xi}.}$$

Then

$$\boldsymbol{\xi} = M^{-1/2}\boldsymbol{y}, \qquad \ddot{\boldsymbol{\xi}} = M^{-1/2}\ddot{\boldsymbol{y}}.$$

Substitute and multiply by $M^{-1/2}$:

$$\boxed{\ddot{\boldsymbol{y}} + D\boldsymbol{y} = \boldsymbol{0}, \qquad D = M^{-1/2} K M^{-1/2}.}$$

Since M and K are symmetric,

$$\boxed{D = D^T.}$$

BOARD Because D is symmetric, choose a complete orthonormal eigenbasis. In component and matrix form,

$$\boxed{D\mathbf{v}_\alpha = \omega_\alpha^2 \mathbf{v}_\alpha} \qquad \Longleftrightarrow \qquad \boxed{DV = V\Omega^2,}$$

where

$$\boxed{V = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_N \end{pmatrix}, \qquad V^T V = I,}$$

and

$$\boxed{\Omega^2 = \operatorname{diag}(\omega_1^2, \ldots, \omega_N^2).}$$

For the nonnegative spectrum used in the modal evolution below, choose $\omega_\alpha \geq 0$ and write

$$\boxed{\Omega = \operatorname{diag}(\omega_1, \ldots, \omega_N).}$$

Negative values of ω_α^2 return in the stability discussion later in the lecture.

BOARD Map the mass-weighted eigenvectors back to the original generalized-coordinate space:

$$\boxed{\boldsymbol{u}_\alpha = M^{-1/2}\mathbf{v}_\alpha} \qquad \Longleftrightarrow \qquad \boxed{U = M^{-1/2}V = \begin{pmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 & \cdots & \boldsymbol{u}_N \end{pmatrix}.}$$

Then the generalized eigenproblem is likewise

$$\boxed{K\boldsymbol{u}_\alpha = \omega_\alpha^2 M \boldsymbol{u}_\alpha} \qquad \Longleftrightarrow \qquad \boxed{KU = MU\Omega^2.}$$

SAY Mass weighting turns the kinetic-energy metric into the identity. The matrix V contains ordinary orthonormal eigenvectors of the symmetric operator D; $U = M^{-1/2}V$ contains the corresponding physical mode shapes.

10–25 min: The correct orthogonality is mass weighted

BOARD Take two generalized modes:

$$K\boldsymbol{u}_\alpha = \omega_\alpha^2 M \boldsymbol{u}_\alpha, \qquad K\boldsymbol{u}_\beta = \omega_\beta^2 M \boldsymbol{u}_\beta.$$

Left-multiply the first by $\boldsymbol{u}_\beta^T$ and the second by $\boldsymbol{u}_\alpha^T$:

$$\boldsymbol{u}_\beta^T K \boldsymbol{u}_\alpha = \omega_\alpha^2 \boldsymbol{u}_\beta^T M \boldsymbol{u}_\alpha,$$

$$\boldsymbol{u}_\alpha^T K \boldsymbol{u}_\beta = \omega_\beta^2 \boldsymbol{u}_\alpha^T M \boldsymbol{u}_\beta.$$

By symmetry of K and M,

$$\boxed{(\omega_\alpha^2 - \omega_\beta^2)\boldsymbol{u}_\beta^T M \boldsymbol{u}_\alpha = 0.}$$

Therefore, for distinct frequencies,

$$\boxed{\boldsymbol{u}_\beta^T M \boldsymbol{u}_\alpha = 0.}$$

SAY The kinetic energy tells us what "perpendicular" means. Euclidean orthogonality is special to coordinates where M is proportional to the identity.

BOARD The matrix construction makes the normalization immediate:

$$U^T M U = V^T M^{-1/2} M M^{-1/2} V = V^T V,$$

so

$$\boxed{\boldsymbol{u}_\alpha^T M \boldsymbol{u}_\beta = \delta_{\alpha\beta}} \qquad \Longleftrightarrow \qquad \boxed{U^T M U = I.}$$

Using $KU = MU\Omega^2$,

$$U^T K U = U^T M U \Omega^2 = \Omega^2,$$

therefore

$$\boxed{\boldsymbol{u}_\alpha^T K \boldsymbol{u}_\beta = \omega_\alpha^2 \delta_{\alpha\beta}} \qquad \Longleftrightarrow \qquad \boxed{U^T K U = \Omega^2.}$$

ASK What changes when two modes have the same frequency?

BOARD If $\omega_\alpha = \omega_\beta$, the component orthogonality proof gives no constraint. But because D is symmetric, we may choose an ordinary orthonormal basis inside each degenerate eigenspace. Thus $V^T V = I$ and therefore $U^T M U = I$ can still be imposed.

25–40 min: Normal coordinates diagonalize both quadratic forms

BOARD Use the modal coordinates. Matrix and component forms are

$$\boxed{\boldsymbol{\xi} = U\boldsymbol{a}} \qquad \Longleftrightarrow \qquad \boxed{\boldsymbol{\xi} = \sum_{\alpha=1}^{N} \boldsymbol{u}_\alpha a_\alpha.}$$

Substitute into the linearized equations and left-multiply by U^T:

$$U^T M U \ddot{\boldsymbol{a}} + U^T K U \boldsymbol{a} = \boldsymbol{0}.$$

Therefore

$$\boxed{\ddot{\boldsymbol{a}} + \Omega^2 \boldsymbol{a} = \boldsymbol{0}} \qquad \Longleftrightarrow \qquad \boxed{\ddot{a}_\alpha + \omega_\alpha^2 a_\alpha = 0.}$$

BOARD The two quadratic forms become

$$\boxed{T^{(2)} = \frac{1}{2}\dot{\boldsymbol{a}}^T \dot{\boldsymbol{a}} = \sum_{\alpha=1}^{N} \frac{1}{2}\dot{a}_\alpha^2,}$$

$$\boxed{V^{(2)} = \frac{1}{2}\boldsymbol{a}^T \Omega^2 \boldsymbol{a} = \sum_{\alpha=1}^{N} \frac{1}{2}\omega_\alpha^2 a_\alpha^2.}$$

Hence

$$\boxed{L^{(2)} = \frac{1}{2}\dot{\boldsymbol{a}}^T \dot{\boldsymbol{a}} - \frac{1}{2}\boldsymbol{a}^T \Omega^2 \boldsymbol{a} = \sum_{\alpha=1}^{N} \left[\frac{1}{2}\dot{a}_\alpha^2 - \frac{1}{2}\omega_\alpha^2 a_\alpha^2\right].}$$

SAY This is the real payoff of the generalized eigenproblem: not merely frequencies, but a change of generalized coordinates that simultaneously simplifies inertia and restoring structure.

BOARD Structural statement:

$$\boxed{N \text{ coupled coordinates} \quad \longrightarrow \quad N \text{ independent quadratic modal coordinates.}}$$

CHECK This is the general version of Lecture 8. There, $M = mI$ and ordinary orthogonality was already the mass-weighted orthogonality up to an overall factor.

40–52 min: Initial conditions, matrix evolution, and zero modes

BOARD Since $\boldsymbol{\xi} = U\boldsymbol{a}$ and $U^T M U = I$,

$$\boxed{\boldsymbol{a} = U^T M \boldsymbol{\xi}} \qquad \Longleftrightarrow \qquad \boxed{a_\alpha = \boldsymbol{u}_\alpha^T M \boldsymbol{\xi}.}$$

Thus the complete initial state projects as

$$\boxed{\boldsymbol{a}(0) = U^T M \boldsymbol{\xi}(0), \qquad \dot{\boldsymbol{a}}(0) = U^T M \dot{\boldsymbol{\xi}}(0),}$$

or componentwise,

$$\boxed{a_\alpha(0) = \boldsymbol{u}_\alpha^T M \boldsymbol{\xi}(0), \qquad \dot{a}_\alpha(0) = \boldsymbol{u}_\alpha^T M \dot{\boldsymbol{\xi}}(0).}$$

SAY Mode decomposition acts on the complete dynamical state, not only on the configuration. We need both modal amplitudes and modal velocities.

BOARD If all frequencies in the block are strictly positive, the component solution and its matrix form are

$$\boxed{a_\alpha(t) = a_\alpha(0)\cos(\omega_\alpha t) + \frac{\dot{a}_\alpha(0)}{\omega_\alpha}\sin(\omega_\alpha t)}$$

$$\boxed{\boldsymbol{a}(t) = \cos(\Omega t)\boldsymbol{a}(0) + \Omega^{-1}\sin(\Omega t)\dot{\boldsymbol{a}}(0), \qquad \Omega > 0.}$$

CHECK The matrix form has the correct initial data because

$$\boldsymbol{a}(0) = I\boldsymbol{a}(0) + 0\dot{\boldsymbol{a}}(0),$$

and

$$\dot{\boldsymbol{a}}(t) = -\Omega\sin(\Omega t)\boldsymbol{a}(0) + \cos(\Omega t)\dot{\boldsymbol{a}}(0),$$

so $\dot{\boldsymbol{a}}(0) = \dot{\boldsymbol{a}}(0)$. Differentiating once more gives

$$\ddot{\boldsymbol{a}}(t) = -\Omega^2\boldsymbol{a}(t).$$

The matrix functions are simple here because Ω is diagonal.

STOP A zero-frequency component instead satisfies

$$\omega_0 = 0 \quad \Rightarrow \quad \ddot{a}_0 = 0, \qquad \boxed{a_0(t) = a_0(0) + \dot{a}_0(0)t.}$$

A zero-frequency mode is not generally static.

BOARD For a nonnegative spectrum, order the columns of U so the positive-frequency modes come first and the zero modes last:

$$\boxed{U = \begin{pmatrix} U_+ & U_0 \end{pmatrix}, \qquad \Omega = \begin{pmatrix} \Omega_+ & 0 \\ 0 & 0 \end{pmatrix}, \qquad \boldsymbol{a} = \begin{pmatrix} \boldsymbol{a}_+ \\ \boldsymbol{a}_0 \end{pmatrix}, \qquad \Omega_+ > 0.}$$

Then the oscillatory and null-space pieces fit into one block propagator:

$$\boxed{\boldsymbol{a}(t) = \begin{pmatrix} \cos(\Omega_+ t) & 0 \\ 0 & I_0 \end{pmatrix}\boldsymbol{a}(0) + \begin{pmatrix} \Omega_+^{-1}\sin(\Omega_+ t) & 0 \\ 0 & tI_0 \end{pmatrix}\dot{\boldsymbol{a}}(0).}$$

Here I_0 is the identity on the zero-mode subspace. Together with

$$\boxed{\boldsymbol{a}(0) = U^T M \boldsymbol{\xi}(0), \qquad \dot{\boldsymbol{a}}(0) = U^T M \dot{\boldsymbol{\xi}}(0), \qquad \boldsymbol{\xi}(t) = U\boldsymbol{a}(t),}$$

this is the complete matrix solution in the original generalized coordinates.

BOARD The same solution in explicit indexed form is

$$\boxed{\boldsymbol{\xi}(t) = \sum_{\omega_\alpha > 0} \boldsymbol{u}_\alpha \left[a_\alpha(0)\cos(\omega_\alpha t) + \frac{\dot{a}_\alpha(0)}{\omega_\alpha}\sin(\omega_\alpha t) \right] + \sum_{\omega_\alpha = 0} \boldsymbol{u}_\alpha \left[a_\alpha(0) + \dot{a}_\alpha(0)t \right].}$$

OPTIONAL Mention only: the velocity block can also be written using the Moore–Penrose pseudoinverse and the projector onto the zero-mode subspace, or as $t\,\mathrm{sinc}(\Omega t)$ with the continuous value at zero. An SVD gives a systematic numerical route to those forms. Do not develop these representations here.

STOP The block form above assumes $\omega_\alpha^2 \geq 0$. Negative eigenvalues of the quadratic restoring form produce exponentially growing/decaying solutions and will be interpreted in the stability discussion later in the lecture.

52–64 min: Degeneracy, symmetry, and basis freedom

BOARD Simplest degenerate example: isotropic two-dimensional oscillator

$$\boxed{M = mI, \qquad K = kI.}$$

Then for every nonzero vector $\boldsymbol{u}$,

$$K\boldsymbol{u} = \frac{k}{m} M\boldsymbol{u}.$$

Thus

$$\boxed{\omega^2 = \frac{k}{m}}$$

for every direction in the plane.

ASK Which pair of eigenvectors should we call "the" two modes?

BOARD We may choose

$$\hat{\mathbf{x}}, \hat{\mathbf{y}},$$

or any rotated orthonormal pair. The eigenspace is physical; a particular basis inside it is not unique.

SAY Rotational symmetry makes all directions equivalent. Degeneracy is therefore natural: symmetry can produce a multidimensional space of motions with the same frequency.

BOARD Matrix version: if U_d contains a mass-orthonormal basis for a degenerate block with frequency ω, then for any orthogonal matrix R,

$$\boxed{U_d \to U_d R, \qquad R^T R = I,}$$

and

$$\boxed{(U_d R)^T M (U_d R) = I, \qquad (U_d R)^T K (U_d R) = \omega^2 I.}$$

The basis changes; the degenerate modal subspace and the reduced dynamics do not.

CHECK Distinguish:

$$\boxed{\text{degeneracy: several independent mode directions share one } \omega}$$

from

$$\boxed{\text{zero mode: } \omega = 0.}$$

A zero mode may itself be degenerate, but the ideas are different.

OPTIONAL Ring callback: the degenerate nonzero-frequency subspace can be represented by traveling-wave modes or by real sine/cosine combinations. The basis changes; the eigenspace does not.

64–72 min: The continuum version of mass-weighted orthogonality

BOARD Finite-dimensional inner product:

$$\boxed{\langle \boldsymbol{u}_\alpha, \boldsymbol{u}_\beta \rangle_M = \boldsymbol{u}_\alpha^T M \boldsymbol{u}_\beta.}$$

SAY In a continuum field, the mass matrix becomes a mass density or weight function.

BOARD For a one-dimensional field with density $\rho(x)$, the corresponding orthogonality has the form

$$\boxed{\int_0^\ell \rho(x) u_\alpha(x) u_\beta(x)\, dx = \delta_{\alpha\beta}}$$

after normalization.

BOARD Spectral expansion:

$$\boxed{\phi(x,t) = \sum_\alpha a_\alpha(t) u_\alpha(x)}$$

for a discrete spectrum.

SAY The matrix normal-mode problem and continuum mode expansions are the same structural move: change from local configuration variables to a basis adapted to the dynamical operator and its kinetic-energy weight.

72–82 min: What the quadratic approximation threw away

BOARD The exact theory near equilibrium contains more than

$$L^{(2)} = \frac{1}{2}\dot{\boldsymbol{\xi}}^T M \dot{\boldsymbol{\xi}} - \frac{1}{2}\boldsymbol{\xi}^T K \boldsymbol{\xi}.$$

Schematically,

$$\boxed{L = L^{(2)} + L^{(3)} + L^{(4)} + \cdots .}$$

BOARD Potential expansion:

$$\boxed{V = V_0 + \frac{1}{2}K_{ij}\xi_i\xi_j + \frac{1}{3!}V^{(3)}_{ijk}\xi_i\xi_j\xi_k + \frac{1}{4!}V^{(4)}_{ijkl}\xi_i\xi_j\xi_k\xi_l + \cdots .}$$

CHECK In general coordinates, $M(\boldsymbol{q})$ also varies with configuration, so higher-order kinetic terms such as

$$\xi\,\dot{\xi}^2$$

can appear. Lecture 11 froze M at equilibrium because those terms are beyond quadratic order.

BOARD Transform to modal coordinates, $\boldsymbol{\xi} = U\boldsymbol{a}$. A cubic potential term becomes schematically

$$\boxed{V^{(3)} = \sum_{\alpha\beta\gamma} C_{\alpha\beta\gamma} a_\alpha a_\beta a_\gamma .}$$

SAY At quadratic order the modes are independent. Beyond quadratic order, the same modal basis is still useful, but the modes can exchange energy and the response can become amplitude dependent.

BOARD Contrast:

linearized theory : superposition + independent modes
nonlinear theory : mode coupling + amplitude dependence

82–98 min: The nonlinear pendulum and the phase portrait

BOARD Exact pendulum:

$$\boxed{\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0.}$$

Energy:

$$\boxed{E = \frac{1}{2}m\ell^2\dot{\theta}^2 + mg\ell(1-\cos\theta).}$$

BOARD Near $\theta = 0$,

$$1 - \cos\theta = \frac{1}{2}\theta^2 - \frac{1}{24}\theta^4 + \cdots ,$$

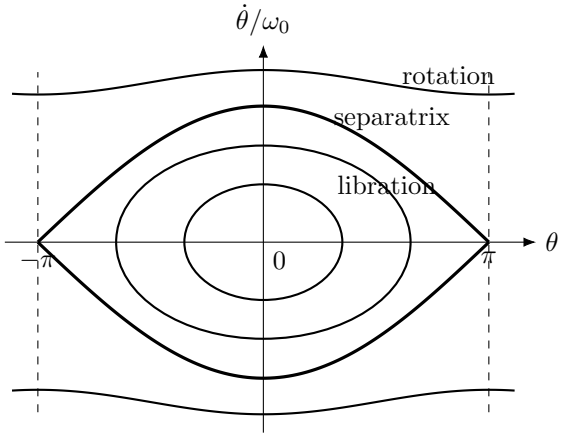
so the quadratic theory gives

$$\boxed{\ddot{\theta} + \frac{g}{\ell}\theta = 0, \qquad \omega_0 = \sqrt{\frac{g}{\ell}}.}$$

ASK What global motions can the exact system have that the local harmonic approximation cannot possibly reveal?

BOARD Solve the energy relation for the phase curves:

$$\boxed{\dot{\theta} = \pm\sqrt{\frac{2}{m\ell^2}\,[E - mg\ell(1-\cos\theta)]}.}$$



BOARD Energy classification:

$$\boxed{0 < E < 2mg\ell \quad \Rightarrow \quad \text{libration: closed phase curves}}$$

$$\boxed{E = 2mg\ell \quad \Rightarrow \quad \text{separatrix through } (\pi, 0)}$$

$$\boxed{E > 2mg\ell \quad \Rightarrow \quad \text{continuous rotation.}}$$

SAY The harmonic approximation captures the local center near $(\theta, \dot\theta) = (0, 0)$. It cannot know that a separatrix exists or that sufficiently energetic trajectories rotate rather than librate.

CHECK The quartic correction is negative:

$$V(\theta) = \frac{1}{2}mg\ell\theta^2 - \frac{1}{24}mg\ell\theta^4 + \cdots .$$

The pendulum therefore softens with increasing amplitude; its period is no longer amplitude independent. No detailed period formula is needed here.

98–105 min: Marginal stability and the final hierarchy

BOARD Compare two potentials near $x = 0$:

$$\boxed{V_+(x) = \frac{\lambda}{4}x^4, \qquad V_-(x) = -\frac{\lambda}{4}x^4, \qquad \lambda > 0.}$$

For both,

$$\left.\frac{dV}{dx}\right|_0 = 0, \qquad \left.\frac{d^2V}{dx^2}\right|_0 = 0.$$

Thus the quadratic stiffness is zero in both cases.

ASK Can the quadratic theory distinguish them?

BOARD No. Yet

$$\boxed{V_+ : \text{ stable minimum,}} \qquad \boxed{V_- : \text{ unstable maximum.}}$$

Higher-order terms decide.

BOARD For a normal mode,

$$\boxed{\omega^2 = \frac{\boldsymbol{u}^T K \boldsymbol{u}}{\boldsymbol{u}^T M \boldsymbol{u}}.}$$

Because M is positive definite, the sign of ω^2 is controlled by the quadratic restoring form.

BOARD Stability hierarchy for a conservative natural system with $M > 0$:

$$\boxed{K \text{ positive definite} \quad \Rightarrow \quad \text{strict quadratic minimum / small oscillations}}$$

K has a negative direction	$\Rightarrow$	some $\omega^2 < 0$ / linear instability

K has a zero direction	$\Rightarrow$	quadratic test inconclusive.

STOP A zero eigenvalue can mean different things:

- an exact continuous symmetry gives a genuinely flat direction and a symmetry zero mode;
- a marginal equilibrium may have zero quadratic curvature but be stabilized or destabilized by higher-order terms.

BOARD Final map:

$$M\ddot{\boldsymbol{\xi}} + K\boldsymbol{\xi} = 0 \to D = M^{-1/2}KM^{-1/2} \to DV = V\Omega^2$$

$$\Downarrow$$

$$U = M^{-1/2}V, \qquad U^T M U = I, \qquad U^T K U = \Omega^2$$

$$\Downarrow$$

$$\boldsymbol{\xi} = U\boldsymbol{a}, \qquad \ddot{\boldsymbol{a}} + \Omega^2\boldsymbol{a} = 0$$

$\Downarrow$ restore omitted terms

nonlinear coupling, amplitude dependence, separatrices, global dynamics

NEXT Next lecture begins rigid-body mechanics. The normal-mode unit leaves us with a general lesson that will recur: choose coordinates and structure that simplify the quadratic forms, but keep track of what information a reduction or approximation has discarded.

If behind: protect the matrix chain $DV = V\Omega^2$, $U = M^{-1/2}V$, $U^TMU = I$, $U^TKU = \Omega^2$, the separate zero-mode time dependence, one symmetry/degeneracy example, the statement that higher-order terms recouple modes, the pendulum separatrix picture, and the marginal-stability conclusion. Shorten the continuum callback, detailed component orthogonality algebra, initial-condition projection formulas, ring callback, and the amplitude-softening remark first.