

Lecture 11 Instructor Outline — Draft

Central Forces II and Small Oscillations I

105 minutes — from exact reduction to local quadratic dynamics

Core arc:

$$\begin{array}{c}
 V_{\text{eff}}(r) \text{ for } V = -k/r \rightarrow \text{bound / escape / scattering} \\
 \Downarrow \\
 u = 1/r \rightarrow r(\theta) = \frac{p}{1 + e \cos \theta} \\
 \Rightarrow \\
 \text{exact reduction by symmetry} \rightarrow \text{local quadratic approximation near equilibrium} \\
 \Downarrow \\
 V^{(2)} = \frac{1}{2} \boldsymbol{\xi}^T K \boldsymbol{\xi}, \quad T^{(2)} = \frac{1}{2} \dot{\boldsymbol{\xi}}^T M \dot{\boldsymbol{\xi}} \\
 \Downarrow \\
 M \ddot{\boldsymbol{\xi}} + K \boldsymbol{\xi} = \mathbf{0} \rightarrow K \mathbf{u} = \omega^2 M \mathbf{u}
 \end{array}$$

Protected goals:

- Use one schematic inverse-square effective-potential plot to classify circular, bound, escape-threshold, and scattering motion before deriving the orbit.
- Recover the conic orbit by eliminating time and using $u(\theta) = 1/r(\theta)$.
- Connect the sign of E to $e < 1$, $e = 1$, $e > 1$; keep detailed scattering formulas optional.
- Make the transition explicit: central-force simplification is exact symmetry reduction; small oscillations are a local quadratic approximation.
- Introduce K as the Hessian of V while reconnecting to the earlier factorization $K = A^T C A$: A is node-to-edge, K node-to-node, and K alone generally loses edge-level realization information.
- Use the resistor-network analogy carefully: $A \mathbf{v}$ gives branch voltage drops; A^T enforces KCL-type node balance; $A^T G A$ is the nodal conductance operator.
- Explain the origin of off-diagonal mass-matrix terms using $M_q = J_q^T M_x J_q$ and one minimal coordinate example.
- Address the natural student question “why not just diagonalize $M^{-1}K$?” without doing the full L12 orthogonality development.
- State separately that a zero-frequency mode has $a_0(t) = a_0(0) + \dot{a}_0(0)t$.

0–8 min: Recover only the essential L10 reduction

BOARD

$$L_{\text{rel}} = \frac{1}{2} \mu (\dot{r}^2 + r^2 \dot{\theta}^2) - V(r)$$

$$J = \mu r^2 \dot{\theta} = \text{constant}$$

$$E = \frac{1}{2} \mu \dot{r}^2 + V_{\text{eff}}(r), \quad V_{\text{eff}}(r) = V(r) + \frac{J^2}{2\mu r^2}$$

SAY Lecture 10 gave an exact radial reduction. Today ask for the geometry: how do we get the curve $r(\theta)$ back?

ASK If the orbit is a geometric curve, why insist on solving for $r(t)$ first?

8–20 min: Inverse-square V_{eff} — read the motion from one picture

BOARD

$$V(r) = -\frac{k}{r}, \quad k > 0$$

so

$$V_{\text{eff}}(r) = -\frac{k}{r} + \frac{J^2}{2\mu r^2}.$$

BOARD Limits for $J \neq 0$:

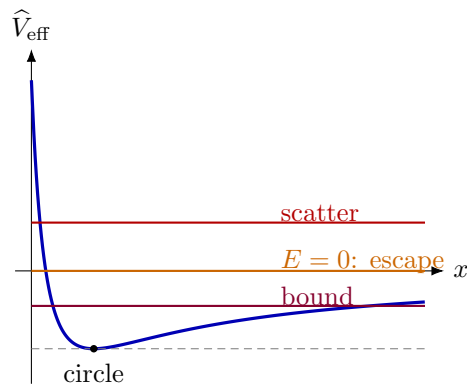
$$V_{\text{eff}} \rightarrow +\infty \quad (r \rightarrow 0^+), \quad V_{\text{eff}} \rightarrow 0^- \quad (r \rightarrow \infty).$$

BOARD Scale with

$$r_c = \frac{J^2}{\mu k}, \quad x = \frac{r}{r_c}, \quad E_0 = \frac{\mu k^2}{2J^2},$$

so

$$\hat{V}_{\text{eff}}(x) = x^{-2} - 2x^{-1}.$$



ASK Without any orbit equation, identify the radial behavior of each horizontal energy line.

BOARD

$$\frac{1}{2}\mu\dot{r}^2 = E - V_{\text{eff}}(r) \implies V_{\text{eff}}(r) \leq E.$$

Turning point:

$$V_{\text{eff}}(r) = E.$$

BOARD Classification:

$E = E_c$	circular orbit
$E_c < E < 0$	two turning points: bound
$E = 0$	escape threshold
$E > 0$	one closest approach + $r \rightarrow \infty$: scattering

SAY This is the mechanics first. The conic derivation should explain and sharpen this picture, not replace it.

20–28 min: Circular orbit and stability

BOARD

$$\frac{dV_{\text{eff}}}{dr} = \frac{k}{r^2} - \frac{J^2}{\mu r^3} = 0$$

so

$$r_c = \frac{J^2}{\mu k}.$$

BOARD

$$E_c = V_{\text{eff}}(r_c) = -\frac{\mu k^2}{2J^2}.$$

BOARD

$$\left. \frac{d^2 V_{\text{eff}}}{dr^2} \right|_{r_c} = \frac{k}{r_c^3} > 0.$$

SAY The circle is the minimum of the reduced one-dimensional potential and is radially stable.

28–42 min: Eliminate time — derive the Binet equation

BOARD

$$\dot{\theta} = \frac{J}{\mu r^2}.$$

Define

$$u(\theta) = \frac{1}{r(\theta)}.$$

BOARD

$$\frac{dr}{d\theta} = -\frac{1}{u^2} \frac{du}{d\theta}$$

therefore

$$\dot{r} = -\frac{J}{\mu} \frac{du}{d\theta}.$$

BOARD Differentiate again:

$$\ddot{r} = -\frac{J^2}{\mu^2} u^2 \frac{d^2 u}{d\theta^2}$$

and

$$r\dot{\theta}^2 = \frac{J^2}{\mu^2} u^3.$$

Hence

$$\ddot{r} - r\dot{\theta}^2 = -\frac{J^2}{\mu^2} u^2 \left(\frac{d^2 u}{d\theta^2} + u \right).$$

BOARD For $F_r = -k/r^2 = -ku^2$,

$$\mu(\ddot{r} - r\dot{\theta}^2) = -ku^2$$

so

$$\frac{d^2 u}{d\theta^2} + u = \frac{\mu k}{J^2}.$$

SAY The inverse-square orbit becomes a shifted harmonic-oscillator equation with angle, not time, as the independent variable.

42–51 min: Conics, energy, and scattering interpretation

BOARD Solution:

$$u(\theta) = \frac{\mu k}{J^2} + A \cos \theta + B \sin \theta.$$

Choose the angular origin along an apsis:

$$u(\theta) = \frac{\mu k}{J^2} (1 + e \cos \theta).$$

Therefore

$$r(\theta) = \frac{p}{1 + e \cos \theta}, \quad p = \frac{J^2}{\mu k}.$$

BOARD

$$E = \frac{\mu k^2}{2J^2}(e^2 - 1)$$

so

$E < 0$	$e < 1$	ellipse / bound
$E = 0$	$e = 1$	parabola / escape threshold
$E > 0$	$e > 1$	hyperbola / scattering

CHECK Return physically to the V_{eff} plot. Same classification from an independent description.

OPTIONAL For $E > 0$, $r \rightarrow \infty$ requires

$$1 + e \cos \theta_\infty = 0, \quad \cos \theta_\infty = -\frac{1}{e}.$$

This identifies asymptotic directions without deriving a full scattering cross section.

51–56 min: Deliberate transition — exact reduction to local approximation

BOARD So far:

$$\text{symmetry} \rightarrow \text{conserved quantities} \rightarrow \text{exact reduction.}$$

ASK What if there is no symmetry strong enough to solve the system exactly?

BOARD New strategy:

$$\text{exact reduction by symmetry} \longrightarrow \text{local quadratic approximation near equilibrium.}$$

SAY “Small” means small displacement from equilibrium, not a small number of degrees of freedom.

56–68 min: Equilibrium and the Hessian stiffness matrix

BOARD Let

$$\mathbf{q} = (q_1, \dots, q_N)^T, \quad \mathbf{q} = \mathbf{q}^{(0)} + \boldsymbol{\xi}.$$

Static equilibrium:

$$\left. \frac{\partial V}{\partial \mathbf{q}} \right|_{\mathbf{q}^{(0)}} = \mathbf{0}.$$

BOARD Expand:

$$V(\mathbf{q}^{(0)} + \boldsymbol{\xi}) = V_0 + \frac{1}{2} \boldsymbol{\xi}^T K \boldsymbol{\xi} + O(\xi^3),$$

where

$$K_{ij} = \left. \frac{\partial^2 V}{\partial q_i \partial q_j} \right|_{\mathbf{q}^{(0)}}.$$

BOARD

$$V^{(2)} = \frac{1}{2} \boldsymbol{\xi}^T K \boldsymbol{\xi}.$$

CHECK Stability:

- K positive definite: quadratic local minimum.
- Negative direction: unstable.
- Zero direction: quadratic test inconclusive unless exact symmetry explains the flat direction.

68–78 min: K is node-to-node; A is node-to-edge

BOARD Recall the more resolved spring-network structure:

$$\mathbf{e} = A\boldsymbol{\xi}, \quad V = \frac{1}{2}\mathbf{e}^T C \mathbf{e} = \frac{1}{2}\boldsymbol{\xi}^T A^T C A \boldsymbol{\xi}.$$

Therefore

$$K = A^T C A.$$

BOARD Keep the first-order structure visible:

$$\boldsymbol{\xi} \xrightarrow{A} \mathbf{e} \xrightarrow{C} C\mathbf{e} \xrightarrow{-A^T} \mathbf{F}_{\text{node}}.$$

SAY A is node-to-edge. K is node-to-node. Knowing A and C gives K , but K generally does not uniquely tell us the edge realization.

BOARD

$$(A, C) \implies K = A^T C A \quad K \not\Rightarrow \text{unique } A, C.$$

SAY If we want only nodal motion, K may be enough. If we want spring extensions, edge forces, local stresses, or load paths, we need the edge-level structure.

BOARD Circuit analogy:

$$\mathbf{v}_{\text{branch}} = A\mathbf{v}, \quad \mathbf{i}_{\text{branch}} = G A \mathbf{v}, \quad A^T \mathbf{i}_{\text{branch}} = \mathbf{i}_{\text{injected}}.$$

Thus

$$A^T G A \mathbf{v} = \mathbf{i}_{\text{injected}}.$$

SAY A^T is KCL-type nodal balance. $A\mathbf{v}$ gives branch voltage drops from node potentials; their loop consistency is the KVL-type side. The nodal operator is a compressed branch-level description.

78–90 min: The mass matrix and why it can be off-diagonal

BOARD General natural kinetic energy:

$$T = \frac{1}{2}\dot{\mathbf{q}}^T M(\mathbf{q})\dot{\mathbf{q}}.$$

Near equilibrium:

$$M(\mathbf{q}) = M(\mathbf{q}^{(0)}) + O(\xi).$$

Since $\dot{\boldsymbol{\xi}} = O(\epsilon)$,

$$\dot{\boldsymbol{\xi}}^T [O(\xi)] \dot{\boldsymbol{\xi}} = O(\epsilon^3),$$

so to quadratic order

$$T^{(2)} = \frac{1}{2}\dot{\boldsymbol{\xi}}^T M \dot{\boldsymbol{\xi}}, \quad M = M(\mathbf{q}^{(0)}).$$

ASK If the particle mass matrix is diagonal, where could off-diagonal terms in M possibly come from?

BOARD Coordinate map:

$$\mathbf{x} = \mathbf{x}(\mathbf{q}), \quad J_q = \frac{\partial \mathbf{x}}{\partial \mathbf{q}^T}, \quad \dot{\mathbf{x}} = J_q \dot{\mathbf{q}}.$$

If

$$T = \frac{1}{2}\dot{\mathbf{x}}^T M_x \dot{\mathbf{x}},$$

then

$$M_q = J_q^T M_x J_q.$$

BOARD Minimal example:

$$x_1 = q_1, \quad x_2 = q_1 + q_2.$$

Then

$$T = \frac{1}{2}m_1\dot{q}_1^2 + \frac{1}{2}m_2(\dot{q}_1 + \dot{q}_2)^2$$

so

$$M = \begin{pmatrix} m_1 + m_2 & m_2 \\ m_2 & m_2 \end{pmatrix}.$$

SAY The off-diagonal term means the two generalized-coordinate motions move some of the same physical mass. Lecture 10's center-of-mass coordinate was a specially chosen coordinate that removed such a mixed term.

90–100 min: Quadratic equations and the generalized eigenproblem

BOARD Quadratic Lagrangian:

$$L^{(2)} = \frac{1}{2}\dot{\boldsymbol{\xi}}^T M \dot{\boldsymbol{\xi}} - \frac{1}{2}\boldsymbol{\xi}^T K \boldsymbol{\xi}.$$

Euler–Lagrange:

$$M\ddot{\boldsymbol{\xi}} + K\boldsymbol{\xi} = \mathbf{0}.$$

BOARD Try

$$\boldsymbol{\xi}(t) = \mathbf{u} a(t)$$

with

$$\ddot{a} + \omega^2 a = 0.$$

Then

$$K\mathbf{u} = \omega^2 M\mathbf{u}.$$

SAY A normal mode is a configuration-space direction for which restoring and inertial response remain proportional, allowing one scalar time dependence.

ASK Why not just multiply by M^{-1} and solve an ordinary eigenproblem?

BOARD Yes, if M is invertible:

$$M^{-1}K\mathbf{u} = \omega^2\mathbf{u}.$$

But keep

$$K\mathbf{u} = \omega^2 M\mathbf{u}$$

because M and K retain distinct mechanical meaning, and $M^{-1}K$ is generally not symmetric in the ordinary Euclidean inner product.

OPTIONAL Preview L12:

$$M^{-1/2}KM^{-1/2}\mathbf{y} = \omega^2\mathbf{y}, \quad \mathbf{y} = M^{1/2}\mathbf{u}.$$

This restores a symmetric ordinary eigenproblem when M is positive definite.

100–105 min: Familiar spring example, zero modes, and handoff

BOARD Two masses between fixed walls:

$$M = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}, \quad K = \kappa \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

Then

$$\kappa \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \mathbf{u} = \omega^2 \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \mathbf{u}.$$

SAY If $m_1 = m_2$, this collapses to the Lecture 8 problem. Unequal inertia is the new feature.

CHECK Zero mode:

$$K\mathbf{u}_0 = 0 \quad \Rightarrow \quad \ddot{a}_0 = 0$$

so

$$a_0(t) = a_0(0) + \dot{a}_0(0)t.$$

Zero frequency does not necessarily mean static.

NEXT Lecture 12 begins from

$$K\mathbf{u}_\alpha = \omega_\alpha^2 M\mathbf{u}_\alpha$$

and develops mass-weighted orthogonality, normal coordinates, degeneracy and symmetry, zero modes, and the limits of the quadratic approximation.

If behind: protect the V_{eff} behavior plot and qualitative classification, the Binet equation and conic result, the exact-to-local transition, the Hessian definition of K , the $K = A^T C A$ information hierarchy, one explicit off-diagonal- M example, and $K\mathbf{u} = \omega^2 M\mathbf{u}$. Shorten the circular-orbit algebra, energy-eccentricity derivation, circuit analogy details, mass-weighting preview, and final two-mass example first.