Lecture 10 Instructor Outline — Draft
Central Forces I: Reduction by Coordinates and Symmetry

105 minutes — working draft

Core arc:

$$(\mathbf{x}_1, \mathbf{x}_2) \text{ with } V = V(|\mathbf{x}_2 - \mathbf{x}_1|)$$
$$\Downarrow$$
$$(\mathbf{R}, \mathbf{r}) \text{ with } \mathbf{r} = \mathbf{x}_2 - \mathbf{x}_1$$
$$\Downarrow$$
$$\text{free center-of-mass motion} + \text{one effective particle of mass } \mu$$
$$\Downarrow$$
$$\text{rotational symmetry} \to \text{conserved angular momentum} \to \text{planar motion}$$
$$\Downarrow$$
$$(r, \theta) \quad \text{with } \theta \text{ cyclic}$$
$$\Downarrow$$
$$\text{one-dimensional radial motion in } V_{\text{eff}}(r) + \text{reconstruction of } \theta(t)$$

Protected goals:

- Begin in the original particle coordinates $\mathbf{x}_1, \mathbf{x}_2$ and write the original kinetic energy explicitly before changing coordinates.
- Motivate $\mathbf{r} = \mathbf{x}_2 - \mathbf{x}_1$ because it simplifies the interaction potential.
- Introduce the center-of-mass coordinate $\mathbf{R}$ as the complementary coordinate that makes the transformation invertible and gives a particularly simple kinetic energy.
- Emphasize that the center-of-mass/relative-coordinate reduction is exact, not an approximation.
- Use L only for the Lagrangian. Use $\mathbf{J}$ for the angular-momentum vector and J for its conserved scalar component in the orbital plane.
- Make the connection to Lecture 7 explicit: translation symmetry removes the center-of-mass motion; rotational symmetry removes the angular degree of freedom from the radial dynamics.
- Treat the $J^2/(2\mu r^2)$ term as angular kinetic energy rewritten using a conserved quantity, not as a new microscopic interaction.
- End with qualitative mechanics from V_{eff}; save Kepler orbit geometry and scattering for Lecture 11.

0–20 min: Choose coordinates that expose the interaction

BOARD Two particles:

$$m_1, \mathbf{x}_1(t), \qquad m_2, \mathbf{x}_2(t).$$

Assume the internal potential depends only on their separation:

$$\boxed{V = V(r), \qquad r = \|\mathbf{x}_2 - \mathbf{x}_1\|.}$$

BOARD Before changing coordinates, write the kinetic energy:

$$\boxed{T = \frac{1}{2} m_1 \dot{\mathbf{x}}_1^2 + \frac{1}{2} m_2 \dot{\mathbf{x}}_2^2.}$$

Hence the original Lagrangian is

$$\boxed{L = \frac{1}{2} m_1 \dot{\mathbf{x}}_1^2 + \frac{1}{2} m_2 \dot{\mathbf{x}}_2^2 - V(|\mathbf{x}_2 - \mathbf{x}_1|).}$$

SAY The kinetic energy naturally knows about the two particles separately. The potential does not: it only knows their separation. Let the interaction tell us the first new coordinate.

BOARD Define the relative coordinate

$$\boxed{\mathbf{r} = \mathbf{x}_2 - \mathbf{x}_1.}$$

Then

$$\boxed{V = V(r), \qquad r = \|\mathbf{r}\|.}$$

ASK One vector coordinate is not enough to replace two vector coordinates. What second coordinate should complete an invertible linear change of variables?

BOARD Define

$$\boxed{M = m_1 + m_2, \qquad \mathbf{R} = \frac{m_1\mathbf{x}_1 + m_2\mathbf{x}_2}{M}.}$$

SAY We are replacing the original particle coordinates $\mathbf{x}_1, \mathbf{x}_2$ by generalized coordinates adapted to the problem. Instead of calling them $\mathbf{q}_1, \mathbf{q}_2$, we give them descriptive names: $\mathbf{R}$ and $\mathbf{r}$.

BOARD The transformation is invertible:

$$\boxed{\mathbf{x}_1 = \mathbf{R} - \frac{m_2}{M}\mathbf{r}, \qquad \mathbf{x}_2 = \mathbf{R} + \frac{m_1}{M}\mathbf{r}.}$$

Differentiate:

$$\boxed{\dot{\mathbf{x}}_1 = \dot{\mathbf{R}} - \frac{m_2}{M}\dot{\mathbf{r}}, \qquad \dot{\mathbf{x}}_2 = \dot{\mathbf{R}} + \frac{m_1}{M}\dot{\mathbf{r}}.}$$

BOARD Substitute these into the *original* kinetic energy:

$$T = \frac{1}{2}m_1\left(\dot{\mathbf{R}} - \frac{m_2}{M}\dot{\mathbf{r}}\right)^2 + \frac{1}{2}m_2\left(\dot{\mathbf{R}} + \frac{m_1}{M}\dot{\mathbf{r}}\right)^2.$$

Expand enough to display the mixed term:

$$\begin{aligned} T = {} & \frac{1}{2}(m_1 + m_2)\dot{\mathbf{R}}^2 \\ & + \left(-\frac{m_1 m_2}{M} + \frac{m_1 m_2}{M}\right)\dot{\mathbf{R}}\cdot\dot{\mathbf{r}} \\ & + \frac{1}{2}\left(\frac{m_1 m_2^2}{M^2} + \frac{m_2 m_1^2}{M^2}\right)\dot{\mathbf{r}}^2. \end{aligned}$$

BOARD Define the reduced mass

$$\boxed{\mu = \frac{m_1 m_2}{M}.}$$

Then

$$\boxed{T = \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2.}$$

SAY Nothing has been approximated. We chose $\mathbf{r}$ to simplify the potential, and $\mathbf{R}$ completes the coordinate transformation in a way that also separates the kinetic energy into overall translation and internal motion.

CHECK Keep the comparison visible:

$$\boxed{\frac{1}{2}m_1\dot{\mathbf{x}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{x}}_2^2} \quad \longrightarrow \quad \boxed{\frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2.}$$

20–32 min: The two-body Lagrangian splits exactly

BOARD Because V depends only on r,

$$\boxed{L = \frac{1}{2}M\dot{\mathbf{R}}^2 + \left[\frac{1}{2}\mu\dot{\mathbf{r}}^2 - V(r)\right].}$$

Write

$$\boxed{L = L_{\rm cm} + L_{\rm rel}}$$

with

$$L_{\rm cm} = \frac{1}{2} M \dot{\mathbf{R}}^{\,2}, \qquad L_{\rm rel} = \frac{1}{2} \mu \dot{\mathbf{r}}^{\,2} - V(r).$$

SAY The original interacting two-particle problem has become a free center-of-mass problem plus the motion of one effective particle of mass μ in the potential $V(r)$.

STOP This is an exact reorganization of the same degrees of freedom, not a heavy-mass approximation and not a small-motion approximation.

BOARD Since

$$\frac{\partial L}{\partial \mathbf{R}} = \mathbf{0},$$

the center-of-mass coordinate is cyclic. Its conjugate momentum is

$$\boxed{\mathbf{P} = \frac{\partial L}{\partial \dot{\mathbf{R}}} = M \dot{\mathbf{R}}.}$$

Euler–Lagrange gives

$$\boxed{\dot{\mathbf{P}} = \mathbf{0}} \qquad \Longrightarrow \qquad \boxed{\ddot{\mathbf{R}} = \mathbf{0}.}$$

SAY Translation symmetry has identified an entire part of the motion that can be solved and set aside. In the center-of-mass frame, $\mathbf{R}$ is constant.

ASK What is left after we remove uniform translation?

BOARD

$$\boxed{L_{\rm rel} = \frac{1}{2} \mu \dot{\mathbf{r}}^{\,2} - V(r).}$$

32–47 min: Central force, angular momentum, and planarity

BOARD The relative equation of motion is

$$\boxed{\mu \ddot{\mathbf{r}} = -\frac{\partial V}{\partial r}\, \hat{\mathbf{r}}, \qquad \hat{\mathbf{r}} = \frac{\mathbf{r}}{r}.}$$

SAY The force is central: it points along the relative position vector. The interaction may be nonlinear and need not be inverse-square.

BOARD Define the relative angular momentum

$$\boxed{\mathbf{J} = \mathbf{r} \times \mu \dot{\mathbf{r}}.}$$

Then

$$\begin{aligned} \dot{\mathbf{J}} &= \dot{\mathbf{r}} \times \mu \dot{\mathbf{r}} + \mathbf{r} \times \mu \ddot{\mathbf{r}} \\ &= \mathbf{0} - \mathbf{r} \times \frac{\partial V}{\partial r} \hat{\mathbf{r}} = \mathbf{0}. \end{aligned}$$

Therefore

$$\boxed{\mathbf{J} = \text{constant}.}$$

SAY This is the same conservation law seen from two viewpoints: rotational symmetry in the Lagrangian, or zero torque for a central force.

BOARD Geometric consequence:

$$\boxed{\mathbf{J} \cdot \mathbf{r} = 0, \qquad \mathbf{J} \cdot \dot{\mathbf{r}} = 0.}$$

For $\mathbf{J} \neq \mathbf{0}$, the relative motion remains in the fixed plane perpendicular to $\mathbf{J}$.

CHECK If $\mathbf{J} = \mathbf{0}$, the motion is purely radial along a fixed line. The planar reduction is then degenerate, not invalid.

47–63 min: Polar coordinates make the symmetry visible

BOARD In the orbital plane use polar generalized coordinates

$$(r, \theta).$$

The velocity decomposes as

$$\boxed{\dot{\mathbf{r}} = \dot{r}\,\hat{\mathbf{r}} + r\dot{\theta}\,\hat{\boldsymbol{\theta}},}$$

so

$$\boxed{\dot{\mathbf{r}}^{\,2} = \dot{r}^2 + r^2\dot{\theta}^2.}$$

Therefore

$$\boxed{L_{\text{rel}} = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r).}$$

BOARD The angle does not appear explicitly:

$$\frac{\partial L_{\text{rel}}}{\partial \theta} = 0.$$

Its conjugate momentum is

$$\boxed{p_\theta = \frac{\partial L_{\text{rel}}}{\partial \dot{\theta}} = \mu r^2\dot{\theta} = J = \text{constant}.}$$

SAY Here J is the conserved signed angular-momentum component normal to the chosen orbital plane. It is the scalar version of $\mathbf{J}$.

CHECK Reinforce the Lecture 7 framing:

$$p_\theta = \frac{\partial L_{\text{rel}}}{\partial \dot{\theta}}, \qquad \dot{p}_\theta = \frac{\partial L_{\text{rel}}}{\partial \theta} = 0.$$

BOARD A short geometric consequence:

$$dA = \frac{1}{2}r^2\,d\theta$$

for the oriented area swept by $\mathbf{r}$, hence

$$\boxed{\dot{A} = \frac{1}{2}r^2\dot{\theta} = \frac{J}{2\mu} = \text{constant}.}$$

SAY Do not need Kepler yet. Equal areas in equal times is a consequence of central force / angular-momentum conservation, not a special property of inverse-square gravity.

63–83 min: Eliminate the cyclic coordinate — the effective potential

BOARD Because L_{rel} has no explicit time dependence,

$$E_{\text{rel}} = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\theta}^2 + V(r)$$

is conserved.

Use

$$\dot{\theta} = \frac{J}{\mu r^2}.$$

Then

$$\frac{1}{2}\mu r^2\dot{\theta}^2 = \frac{J^2}{2\mu r^2}.$$

Therefore

$$\boxed{E_{\text{rel}} = \frac{1}{2}\mu\dot{r}^2 + V_{\text{eff}}(r),}$$

where

$$\boxed{V_{\text{eff}}(r) = V(r) + \frac{J^2}{2\mu r^2}.}$$

STOP The term

$$\frac{J^2}{2\mu r^2}$$

is not a new microscopic interaction. It is the angular kinetic energy rewritten as a function of r after using the conserved value of J.

BOARD Derive the radial equation directly from Euler–Lagrange:

$$\frac{\partial L_{\text{rel}}}{\partial \dot{r}} = \mu \dot{r},$$

so

$$\frac{d}{dt}\left(\frac{\partial L_{\text{rel}}}{\partial \dot{r}}\right) = \mu \ddot{r}.$$

Also

$$\frac{\partial L_{\text{rel}}}{\partial r} = \mu r \dot{\theta}^2 - \frac{\partial V}{\partial r}.$$

Thus

$$\boxed{\mu \ddot{r} = \mu r \dot{\theta}^2 - \frac{\partial V}{\partial r}.}$$

Eliminate $\dot{\theta}$:

$$\boxed{\mu \ddot{r} = \frac{J^2}{\mu r^3} - \frac{\partial V}{\partial r}.}$$

But

$$\frac{dV_{\text{eff}}}{dr} = \frac{\partial V}{\partial r} - \frac{J^2}{\mu r^3},$$

so

$$\boxed{\mu \ddot{r} = -\frac{dV_{\text{eff}}}{dr}.}$$

SAY After specifying the conserved J, the radial coordinate obeys an ordinary one-dimensional mechanics problem. The angle is reconstructed afterward from

$$\boxed{\dot{\theta} = \frac{J}{\mu r^2}.}$$

BOARD Structural reduction:

two particles in 3D $\rightarrow$ relative particle in 3D $\rightarrow$ motion in a plane

$\Downarrow$

one radial degree of freedom + one quadrature

83–96 min: Read the mechanics from the effective potential

BOARD For fixed E_{rel} and J,

$$\frac{1}{2}\mu \dot{r}^2 = E_{\text{rel}} - V_{\text{eff}}(r).$$

Therefore the dynamically allowed region satisfies

$$\boxed{V_{\text{eff}}(r) \leq E_{\text{rel}}.}$$

Turning points:

$$\boxed{\dot{r} = 0 \quad \Longleftrightarrow \quad V_{\text{eff}}(r) = E_{\text{rel}}.}$$

Circular relative motion:

$$\boxed{\frac{dV_{\text{eff}}}{dr} = 0.}$$

Radial stability of a circular orbit:

$$\boxed{\frac{d^2V_{\text{eff}}}{dr^2} > 0 \quad \text{at the circular radius.}}$$

ASK What does nonzero angular momentum do as $r \to 0$?

BOARD If $J \neq 0$,

$$\frac{J^2}{2\mu r^2} \to +\infty \qquad (r \to 0^+).$$

SAY This is often called the angular-momentum or centrifugal barrier. Again: it is a feature of the reduced radial description, not an added interparticle force.

96–105 min: Concrete central spring and handoff

BOARD Take a central spring interaction

$$\boxed{V(r) = \frac{1}{2}K(r - r_0)^2.}$$

Then

$$\boxed{V_{\text{eff}}(r) = \frac{1}{2}K(r - r_0)^2 + \frac{J^2}{2\mu r^2}.}$$

For $J \neq 0$,

$$V_{\text{eff}} \to +\infty \quad \text{as } r \to 0^+ \quad \text{and as } r \to \infty.$$

Therefore there is at least one minimum. A circular orbit satisfies

$$\boxed{K(r_* - r_0) = \frac{J^2}{\mu r_*^3}.}$$

And

$$\boxed{\frac{d^2V_{\text{eff}}}{dr^2} = K + \frac{3J^2}{\mu r^4} > 0,}$$

so the circular relative motion is radially stable.

OPTIONAL The special case $r_0 = 0$ is the isotropic harmonic central potential

$$V(r) = \frac{1}{2}Kr^2.$$

Its full vector equation can be solved exactly, and nonzero-angular-momentum trajectories are ellipses centered on the force center. This is a useful homework or student-notes extension, not required for today's core reduction.

BOARD Final conceptual map:

interaction suggests $\mathbf{r}$
$\downarrow$
$\mathbf{R}, \mathbf{r}$ separate overall and internal motion
$\downarrow$
translation symmetry $\to \mathbf{P}$
$\downarrow$
rotation symmetry $\to \mathbf{J}$
$\downarrow$
θ cyclic $\to J$
$\downarrow$
$V_{\text{eff}}(r)$ organizes radial motion

NEXT Next lecture specialize the machinery rather than rebuild it. For the inverse-square interaction,

$$V(r) = -\frac{k}{r},$$

use the effective potential and angular-momentum structure to develop orbital geometry, the Kepler problem, and scattering.

If behind: protect the explicit coordinate change and kinetic-energy separation, the exact split into $L_{\text{cm}} + L_{\text{rel}}$, angular-momentum conservation and planarity, the cyclic-coordinate derivation of J, and the construction/interpretation of V_{eff}. Shorten the areal-velocity discussion, the detailed radial Euler–Lagrange derivation, and the central-spring example first.