Graduate Classical Mechanics
Lecture 3 Instructor Outline

Newtonian Mechanics, Vectors, and the Structure of Force

Lecture goal: Give a compact Newtonian review that establishes the notation and mathematical language needed before action. Distinguish vectors from their representations, briefly expose the geometric-algebra viewpoint, connect Newtonian state evolution to Lecture 1, derive conservative potential energy from exactness of the work form, briefly expose the geometric-product split of power/turning and virial/torque, connect the Hessian of V to $A^{\mathsf{T}}CA$ from Lecture 2, and end with constraints as the reason to introduce generalized coordinates.

Notation convention to enforce throughout:

$$\boxed{\text{Newton: } (\boldsymbol{x}, \dot{\boldsymbol{x}}) \qquad \text{Lagrange: } (\boldsymbol{q}, \dot{\boldsymbol{q}}) \qquad \text{Hamilton: } (\boldsymbol{q}, \boldsymbol{p})}$$

q is a generalized x and may equal x in simple systems.

Suggested timing for a 105-minute meeting:

- 0–20 min: vectors, basis, inner product;
- 20–30 min: geometric-algebra aside;
- 30–55 min: Newtonian state, first-order form, and explicit callback to $S_{n+1} = F(S_n)$;
- 55–76 min: work, exactness, potential, energy, brief geometric-product aside;
- 76–89 min: equilibrium, Hessian, $A^{\mathsf{T}}CA$;
- 89–100 min: constraints and generalized coordinates;
- 100–105 min: transition to discrete action.

0. Re-entry from Lecture 2

SAY: We have spent two lectures asking where quantities live and what the operators know about the boundaries. Before we replace force balance by an action principle, make sure our Newtonian language is precise.

BOARD:

$$\boxed{M\ddot{\boldsymbol{u}} + A^{\mathsf{T}}CA\boldsymbol{u} = \boldsymbol{f}.}$$

ASK: What part is kinematics? What part is material/force information? What variable is the actual Newtonian configuration?

1. What is a vector?

BOARD:

$$\boxed{\text{vector} = \text{something you can add and scale like real numbers.}}$$

SAY: Two common descriptions are incomplete:

- "magnitude and direction" already assumes an inner product;
- "a list of numbers" is a coordinate representation in a chosen basis.

BOARD:

$$\boldsymbol{v} = v^i \boldsymbol{e}_i.$$

ASK: If I rotate/change the basis, what changes: the vector or the numbers v^i?

BOARD:

$$\|\boldsymbol{v}\|^2 = \boldsymbol{v} \cdot \boldsymbol{v}, \qquad \boldsymbol{u} \cdot \boldsymbol{v} = \|\boldsymbol{u}\|\|\boldsymbol{v}\| \cos\theta.$$

SAY: Inner product is extra structure. It is what gives length and angle.

OPTIONAL: Points versus vectors:

$$Q - P = \text{displacement vector}, \qquad \boldsymbol{x} = P - O.$$

The origin is part of the representation.

2. Very brief geometric-algebra aside

BOARD:

$$\boxed{\boldsymbol{u}\boldsymbol{v} = \boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \wedge \boldsymbol{v}.}$$

SAY: We are not switching formalisms. I only want you to know that the oriented plane spanned by two vectors is itself a natural geometric object—a bivector.

BOARD:

$$\boldsymbol{\tau} \sim \boldsymbol{x} \wedge \boldsymbol{F}, \qquad \vec{\tau} = \boldsymbol{x} \times \boldsymbol{F} \quad \text{in 3D representation.}$$

STOP: If the vector discussion runs long, stop the GA material here. Return to it with rigid bodies.

3. Newtonian state and equations

BOARD:

$$\boxed{m\ddot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x}, \dot{\boldsymbol{x}}, t).}$$

ASK: Is $\boldsymbol{x}$ alone a complete state?

Expected: no; same position can have different velocities and futures.

BOARD:

$$\boxed{S_N = (\boldsymbol{x}, \dot{\boldsymbol{x}}).}$$

BOARD: Convert second order to first order:

$$\boxed{\frac{\mathrm{d}S}{\mathrm{d}t} = f(S, t) = \begin{pmatrix} \dot{\boldsymbol{x}} \\ m^{-1}\boldsymbol{F}(\boldsymbol{x}, \dot{\boldsymbol{x}}, t) \end{pmatrix}.}$$

SAY: Do not stop here. This is the differential form of the Lecture 1 picture. Go all the way back to the finite map.

BOARD:

$$\boxed{S_{n+1} = F(S_n).}$$

SAY: There are two logically distinct readings. The equation above does not by itself decide whether the underlying world is discrete or continuous.

BOARD: *1. Finite update fundamental:*

$$S_{n+1} = F(S_n), \qquad \dot{S}\Big|_{t_n} \approx \frac{S_{n+1} - S_n}{\Delta t_n}.$$

Hence a smooth approximation gives

$$\boxed{S_{n+1} \approx S_n + \Delta t_n\, f(S_n, t_n).}$$

SAY: Here the derivative is the compressed/local approximation to finite change.

BOARD: *2. Continuous evolution fundamental:*

$$\dot{S} = f(S, t)$$

implies exactly

$$S(t_{n+1}) = S(t_n) + \int_{t_n}^{t_{n+1}} f(S(t), t)\,\mathrm{d}t.$$

For autonomous dynamics,

$$\boxed{S_{n+1} = \Phi_{\Delta t}(S_n) \equiv F(S_n).}$$

CHECK: The exact flow map $\Phi_{\Delta t}$ is not forward Euler. Only

$$\Phi_{\Delta t}(S_n) \approx S_n + \Delta t\, f(S_n)$$

is the numerical approximation.

ASK: So what survives in either viewpoint?

Expected:

$$\boxed{\text{complete state} + \text{evolution rule} \Rightarrow \text{next state.}}$$

OPTIONAL: Explicit time dependence: either write

$$S_{n+1} = F_n(S_n)$$

or enlarge the state,

$$\widetilde{S} = (\boldsymbol{x}, \dot{\boldsymbol{x}}, t), \qquad \dot{\widetilde{S}} = \begin{pmatrix} \dot{\boldsymbol{x}} \\ m^{-1}\boldsymbol{F}(\boldsymbol{x}, \dot{\boldsymbol{x}}, t) \\ 1 \end{pmatrix},$$

so that $\widetilde{S}_{n+1} = \widetilde{F}(\widetilde{S}_n)$ is autonomous.

STOP: If time is tight, do the two boxed arrows and the exact-flow-versus-Euler check; leave the explicit-time enlargement for the student notes.

BOARD: Notation roadmap:

$$\boxed{\boldsymbol{x}, \dot{\boldsymbol{x}} \;\longrightarrow\; \boldsymbol{q}, \dot{\boldsymbol{q}} \;\longrightarrow\; \boldsymbol{q}, \boldsymbol{p}}$$

Newton $\rightarrow$ Lagrange $\rightarrow$ Hamilton.

4. Many particles: internal forces cancel

BOARD:

$$m_n \ddot{\boldsymbol{x}}_n = \sum_{k \neq n} \boldsymbol{F}^{\text{int}}_{nk} + \boldsymbol{F}^{\text{ext}}_n.$$

BOARD: Weak third law:

$$\boldsymbol{F}^{\text{int}}_{nk} = -\boldsymbol{F}^{\text{int}}_{kn}.$$

BOARD:

$$\boldsymbol{X}_{\text{cm}} = \frac{1}{M} \sum_n m_n \boldsymbol{x}_n, \qquad \boxed{M \ddot{\boldsymbol{X}}_{\text{cm}} = \sum_n \boldsymbol{F}^{\text{ext}}_n.}$$

SAY: Connect to Lecture 2: internal spring forces cancel in the whole-system balance for the same structural reason.

OPTIONAL: Angular momentum if useful:

$$\boldsymbol{L} = \sum_n (\boldsymbol{x}_n - \boldsymbol{x}_0) \times m_n \dot{\boldsymbol{x}}_n, \qquad \dot{\boldsymbol{L}} = \boldsymbol{\tau}_{\text{ext}}$$

when pair forces are central.

5. Work, exactness, energy, and potential

BOARD: Start from Newton:

$$m\ddot{\boldsymbol{x}} = \boldsymbol{F}.$$

Contract with $\dot{\boldsymbol{x}}$:

$$\boldsymbol{F} \cdot \dot{\boldsymbol{x}} = m\ddot{\boldsymbol{x}} \cdot \dot{\boldsymbol{x}} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2} m \dot{\boldsymbol{x}}^2\right).$$

BOARD:

$$\boxed{\dot{T} = \boldsymbol{F} \cdot \dot{\boldsymbol{x}}}, \qquad T = \frac{1}{2} m \dot{\boldsymbol{x}}^2.$$

SAY: Do not introduce T as a memorized formula. We notice an exact derivative on the inertial side of Newton's equation.

BOARD: Along the trajectory,

$$\mathrm{d}T = \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{x}.$$

ASK: Now make the symmetric move: when is the force side also an exact differential of a scalar function of configuration?

BOARD: For a position-dependent force define the work form

$$\omega = \boldsymbol{F}(\boldsymbol{x}) \cdot \mathrm{d}\boldsymbol{x}.$$

If it is exact,

$$\boxed{\omega = -\mathrm{d}V.}$$

Since $\mathrm{d}V = \nabla V \cdot \mathrm{d}\boldsymbol{x}$,

$$\boxed{\boldsymbol{F} = -\nabla V.}$$

SAY: This is the definition of conservative structure in this language: the work one-form is exact. We did not need to posit V first.

BOARD: The two exact derivatives now meet:

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \boldsymbol{F}\cdot\dot{\boldsymbol{x}} = -\frac{\mathrm{d}V}{\mathrm{d}t},$$

so

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}(T+V) = 0.}$$

BOARD: Boundary/path form:

$$\int_A^B \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{x} = -[V(B)-V(A)], \qquad \oint \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{x} = 0.$$

SAY: Callback to Lecture 2: accumulated local change becomes boundary information. In a simply connected 3-D region this is equivalent to $\nabla\times\boldsymbol{F} = 0$.

CHECK: If $V = V(\boldsymbol{x}, t)$, do not claim energy conservation automatically:

$$\frac{\mathrm{d}}{\mathrm{d}t}(T+V) = \frac{\partial V}{\partial t}$$

for the conservative force alone; add $\boldsymbol{F}_{\mathrm{nc}}\cdot\dot{\boldsymbol{x}}$ if nonconservative forces act.

OPTIONAL: Geometric-product payoff from the earlier aside:

$$\dot{\boldsymbol{x}}\boldsymbol{F} = \dot{\boldsymbol{x}}\cdot\boldsymbol{F} + \dot{\boldsymbol{x}}\wedge\boldsymbol{F}.$$

Scalar part = power; bivector part = transverse/turning action. A normal force can have zero scalar part and nonzero bivector part.

BOARD: If time permits,

$$\boldsymbol{x}\boldsymbol{F} = \boldsymbol{x}\cdot\boldsymbol{F} + \boldsymbol{x}\wedge\boldsymbol{F}.$$

SAY: The bivector part is torque; the scalar part feeds the virial identity. Point interested students to the Lecture 3 geometric-product supplement.

6. Equilibrium and linearization: connect directly to Lecture 2

BOARD: Equilibrium:

$$\nabla V\big|_{\boldsymbol{x}^{(0)}} = 0, \qquad \boldsymbol{u} = \boldsymbol{x} - \boldsymbol{x}^{(0)}.$$

BOARD: Taylor expansion:

$$V(\boldsymbol{x}^{(0)}+\boldsymbol{u}) \simeq V_0 + \frac{1}{2}\boldsymbol{u}^{\mathsf{T}} H\boldsymbol{u},$$

$$H_{ij} = \left.\frac{\partial^2 V}{\partial x_i \partial x_j}\right|_{\boldsymbol{x}^{(0)}}.$$

BOARD:

$$\boxed{\boldsymbol{F}_{\mathrm{int}} \simeq -H\boldsymbol{u}.}$$

ASK: What is H for our spring chain?

BOARD:

$$\boxed{H = A^{\mathsf{T}}CA.}$$

Therefore

$$\boxed{M\ddot{\boldsymbol{u}} + A^{\mathsf{T}}CA\boldsymbol{u} = \boldsymbol{f}.}$$

SAY: Save eigenvalues. At this point the important statement is: stiffness matrix = Hessian of the potential.

7. Constraints: why Newton becomes awkward

BOARD:

$$f(\boldsymbol{x}) = 0, \qquad m\ddot{\boldsymbol{x}} = \boldsymbol{F}_{\text{applied}} + \boldsymbol{F}_{\text{constraint}}.$$

SAY: The constraint force is often an unknown we do not care about.

BOARD: Double-pendulum counting:

$$(x_1, y_1, x_2, y_2) \quad \text{but two rigid-length constraints.}$$

Thus only two independent degrees of freedom.

BOARD:

$$\boxed{\boldsymbol{q} = (\theta_1, \theta_2), \qquad \boldsymbol{x}_n = \boldsymbol{x}_n(\boldsymbol{q}, t).}$$

SAY: This is where q begins: a generalized version of x adapted to the allowed configuration space.

ASK: What would Newton require in Cartesian coordinates? Answer: equations for x_i, y_i plus unknown constraint forces/reactions and constraint equations.

8. Close: why action next?

BOARD:

$$\boxed{\text{Newton: vector force law} \quad \longrightarrow \quad \text{Lagrange: scalar } L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)}$$

SAY: We have already seen one scalar compression, $\boldsymbol{F} = -\nabla V$. The next question is whether a scalar attached to an entire path can generate the dynamics.

NEXT: Start next lecture from a finite path

$$\boldsymbol{q}_0, \boldsymbol{q}_1, \ldots, \boldsymbol{q}_N$$

and construct the discrete action before taking a continuum limit.