Graduate Classical Mechanics
Lecture 2 Instructor Outline

Discrete Mechanics—Chains, Rings, Forces, and Boundaries

Lecture goal: Turn the node/edge philosophy of Lecture 1 into an explicit mechanical system. Derive the spring-chain stiffness/load operator $A^{\mathsf{T}}CA$ and distinguish it from the internal force $-A^{\mathsf{T}}CA$, keep the boundary visible, contrast the open chain with the ring, and make boundary conditions physically concrete.

Likely board content: about 6–7 boards/panels. The discrete integration-by-parts derivation is the longest board calculation. If time is short, save the general edge-field/winding discussion for the notes.

0. Re-entry from Lecture 1

BOARD:

$$\boxed{\text{state on nodes} \qquad \text{first change on edges}}$$

SAY: Last time this was a conceptual picture. Today we make it mechanics.

ASK: If q_i is the displacement of a mass, what physical quantity naturally belongs to the edge between masses i and $i+1$?

Expected: spring extension; then tension/force associated with the spring.

1. Fixed–fixed chain: make the boundary explicit

BOARD: Draw five nodes. Nodes 0 and 4 are fixed; masses M_1, M_2, M_3 move; four springs have constants $c_1, \ldots, c_4$.

BOARD:

$$\boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \qquad u_0 = u_4 = 0.$$

BOARD: Spring extensions:

$$\boldsymbol{e} = \begin{pmatrix} u_1 \\ u_2 - u_1 \\ u_3 - u_2 \\ -u_3 \end{pmatrix} = A\boldsymbol{u},$$

$$\boxed{A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}}.$$

ASK: What do the first and last rows mean physically?

Expected: the first and last springs terminate on fixed supports, so their extensions are u_1 and $-u_3$.

SAY: The boundary conditions are not added later. They are already encoded in the shape and entries of A: four edges, three free node variables.

CHECK:

$$\operatorname{rank}(A) = 3.$$

No nullspace here. Save translational zero modes for the free chain/ring.

OPTIONAL: Full-incidence explanation:

$$\widetilde{A} = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}, \qquad A = \widetilde{A}B,$$

where B embeds (u_1, u_2, u_3) into $(0, u_1, u_2, u_3, 0)$.

2. Energy, constitutive law, and the sign convention

BOARD:

$$\boldsymbol{T} = C\boldsymbol{e} = CA\boldsymbol{u},$$
$$U(\boldsymbol{u}) = \frac{1}{2}\boldsymbol{e}^{\mathsf{T}}C\boldsymbol{e} = \frac{1}{2}\boldsymbol{u}^{\mathsf{T}}A^{\mathsf{T}}CA\boldsymbol{u}.$$

BOARD:

$$\boxed{\nabla_{\boldsymbol{u}}U = A^{\mathsf{T}}CA\boldsymbol{u}.}$$

SAY: In static equilibrium,

$$\boxed{A^{\mathsf{T}}CA\boldsymbol{u} = \boldsymbol{f}}$$

is the external load required to hold the displacement.

CHECK: Internal spring force:

$$\boxed{\boldsymbol{F}_{\text{int}} = -A^{\mathsf{T}}CA\boldsymbol{u}.}$$

BOARD: Structural picture:

$$\boxed{\boldsymbol{u} \xrightarrow{A} \boldsymbol{e} \xrightarrow{C} \boldsymbol{T} \xrightarrow{A^{\mathsf{T}}} \boldsymbol{f}}$$

3. Explicit fixed-boundary stiffness matrix

BOARD:

$$A^{\mathsf{T}}CA = \begin{pmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 + c_4 \end{pmatrix}.$$

For $c_i = c$,

BOARD:

$$\boxed{A^{\mathsf{T}}CA = c\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.}$$

SAY: Every moving mass has two springs. The boundary is encoded by the wall connections.

CHECK: Positive definite for $c_i > 0$ because A has full column rank.

4. Newton equation and continuum preview

BOARD:

$$\boxed{M\ddot{\boldsymbol{u}} + A^{\mathsf{T}}CA\boldsymbol{u} = \boldsymbol{f}.}$$

For identical m, c:

BOARD:

$$m\ddot{\boldsymbol{u}} + c\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\boldsymbol{u} = \boldsymbol{f}.$$

SAY: This is exact for the finite spring system, not yet an approximation.

BOARD: Continuum correspondence:

$$A = \frac{\mathrm{d}}{\mathrm{d}x}, \qquad A^{\mathsf{T}} = -\frac{\mathrm{d}}{\mathrm{d}x}, \qquad \boxed{A^{\mathsf{T}}CAu = -\frac{\mathrm{d}}{\mathrm{d}x}\left(C\frac{\mathrm{d}u}{\mathrm{d}x}\right)}.$$

SAY: This is why the positive operator $A^{\mathsf{T}}CA$ is the natural convention.

5. Boundary data: node versus edge

BOARD:

$$\boxed{u_0 = \text{given}} \qquad \text{vs.} \qquad \boxed{T_{1/2} = \text{given}}.$$

SAY: These are different geometric kinds of boundary information: node value versus edge/flux value.

BOARD:

- fixed end $\to$ prescribe u;
- applied boundary force $\to$ prescribe T;
- endpoint spring $\to$ relation between u and T.

SAY: Later language: Dirichlet, Neumann, Robin. Introduce the names only after the physics is clear.

CHECK: Important subtlety: a finite endpoint mass that is allowed to move is a dynamical degree of freedom, not a "free boundary condition." Its equation is

$$m_0\ddot{u}_0 = T_{1/2} + f_0.$$

Zero boundary tension corresponds to a massless/unloaded free field endpoint or the natural condition of a free variational boundary.

6. Discrete FTC and integration by parts

SAY: To expose a general boundary term, temporarily use the full incidence matrix $\widetilde{A}$ before fixing endpoint displacements.

BOARD: Define point-supported basis vectors:

$$\boxed{(\mathbf{e}^{(j)})_i = \delta_{ij}.}$$

Let $\mathbf{1}_E$ be the all-ones vector in edge space.

BOARD:

$$\boxed{\widetilde{A}^{\mathsf{T}}\mathbf{1}_E = -\mathbf{e}^{(0)} + \mathbf{e}^{(N)}.}$$

Therefore

BOARD:

$$\begin{aligned}\mathbf{1}_E^{\mathsf{T}}\widetilde{A}\widetilde{\boldsymbol{u}} &= (\widetilde{A}^{\mathsf{T}}\mathbf{1}_E)^{\mathsf{T}}\widetilde{\boldsymbol{u}}\\ &= u_N - u_0.\end{aligned}$$

SAY: This is the discrete FTC: sum the local differences, get the oriented boundary.

BOARD: For the reduced fixed–fixed matrix $A = \widetilde{A}B$,

$$A^{\mathsf{T}}\mathbf{1}_E = 0,$$

because the boundary variations have already been removed.

BOARD: General edge vector $\boldsymbol{v}$:

$$\boxed{\boldsymbol{v}^{\mathsf{T}}\widetilde{A}\widetilde{\boldsymbol{u}} = \widetilde{\boldsymbol{u}}^{\mathsf{T}}\widetilde{A}^{\mathsf{T}}\boldsymbol{v}.}$$

BOARD: Expand only after the matrix identity:

$$\begin{aligned}\sum_{i=0}^{N-1} v_{i+1/2}(u_{i+1} - u_i) = -&\sum_{i=1}^{N-1} u_i(v_{i+1/2} - v_{i-1/2})\\ &+ u_N v_{N-1/2} - u_0 v_{1/2}.\end{aligned}$$

SAY: The matrix identity is the structure; the indexed formula is the familiar summation-by-parts expansion.

7. Ring: same local law, different topology

BOARD: Draw same 4 nodes as a ring. Orient $0 \to 1 \to 2 \to 3 \to 0$.

BOARD:

$$A_{\text{ring}} = \begin{pmatrix} -1 & 1 & 0 & 0\\ 0 & -1 & 1 & 0\\ 0 & 0 & -1 & 1\\ 1 & 0 & 0 & -1 \end{pmatrix}.$$

BOARD:

$$A_{\text{ring}}^{\mathsf{T}}A_{\text{ring}} = \begin{pmatrix} 2 & -1 & 0 & -1\\ -1 & 2 & -1 & 0\\ 0 & -1 & 2 & -1\\ -1 & 0 & -1 & 2 \end{pmatrix}.$$

SAY: Now every node has the same local stencil. No fixed supports and no boundary rows.

BOARD:

$$\boxed{A_{\text{ring}}\mathbf{1} = 0.}$$

SAY: This is where the translational nullspace belongs physically: a rigid translation is allowed for the ring, but not for the fixed–fixed chain.

BOARD:

$$\sum_{i=0}^{N-1}(u_{i+1}-u_i)=0, \qquad u_N=u_0.$$

SAY: Same FTC principle as last time: accumulated exact difference equals boundary data; the ring has no boundary.

BOARD:

$$\boxed{\text{chain teaches boundaries; ring teaches cycles.}}$$

8. Internal forces and support reactions

BOARD: Ring:

$$\begin{aligned}\mathbf{1}^\top \boldsymbol{F}_{\text{int}} &= -\mathbf{1}^\top A_{\text{ring}}^\top \boldsymbol{T}\\ &= -(A_{\text{ring}}\mathbf{1})^\top \boldsymbol{T}\\ &= 0.\end{aligned}$$

SAY: For a closed/free system, pair forces cancel in the total.

BOARD: Fixed–fixed chain:

$$A\mathbf{1}=\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}\neq 0.$$

ASK: Does this violate Newton's third law?

Expected: no. The reduced model omits the fixed supports; wall reactions are external to the three-mass subsystem.

SAY: Include the supports and the total pair-force sum is zero again.

9. Optional global edge-field point

OPTIONAL: Use only if time or if the class is engaged with the topology point.

BOARD: If

$$a_{i+1/2}=u_{i+1}-u_i$$

on a ring, then necessarily

$$\boxed{\sum_{\text{ring}} a_{i+1/2}=0.}$$

ASK: Is every assignment of edge values therefore the difference of a globally defined node value?

Expected: no; nonzero circulation obstructs it.

OPTIONAL: Angular variable:

$$\sum_i \Delta\theta_{i+1/2}=2\pi m.$$

Mention winding, but do not develop topology formally.

10. Recommended computational demo: static chain

OPTIONAL: Use if time permits. Save eigenvalues and normal modes for the later oscillations lecture.

SAY: Solve the static boundary-value problem

$$\boxed{A^{\mathsf{T}} C A \boldsymbol{u} = \boldsymbol{f}}$$

for a long fixed–fixed chain with a constant applied load.

BOARD: For unit springs and unit load,

$$2u_i - u_{i-1} - u_{i+1} = 1, \qquad u_0 = u_{M+1} = 0.$$

ASK: What kind of function has constant second difference?

Expected: a quadratic.

BOARD:

$$\boxed{u_i = \frac{i(M+1-i)}{2}.}$$

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% M free masses, M+1 springs, and two fixed endpoints
M = 30;
N = M + 1;

% Reduced incidence matrix: (M+1)-by-M
A = toeplitz([1 -1 zeros(1,N-2)], ...
             [1 zeros(1,M-1)]);

q = 0:M+1; % all node positions, including boundaries
f = ones(M,1); % constant unit load on free masses
C = eye(N,N); % unit spring constants

% Static solve: A'*C*A*u_free = f
u = [0; (A'*C*A)\f; 0];

% Exact quadratic at the nodes
u_exact = -q.*(q-(M+1))/2;

plot(q,u,'o',q,u_exact,'-');
xlabel('node position');
ylabel('displacement');
legend('discrete solution','quadratic solution','Location','best');
grid on;
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CHECK: The points from the matrix solve lie exactly on the parabola.

SAY: This is not merely a continuum approximation in this example: a quadratic has exactly constant second difference, so it solves the discrete equations at the nodes.

ASK: Where are the fixed endpoints encoded?

Expected: in the first and last rows of the rectangular A; the boundary displacements are not unknowns.

11. Closing synthesis

BOARD:

$$\boxed{\boldsymbol{u} \xrightarrow{A} \boldsymbol{e} \xrightarrow{C} \boldsymbol{T} \xrightarrow{A^{\mathsf{T}}} \boldsymbol{f}}$$

BOARD:

$$\boxed{M\ddot{\boldsymbol{u}} + A^{\mathsf{T}} C A \boldsymbol{u} = \boldsymbol{f}}$$

CHECK: Internal spring force:

$$\boxed{\boldsymbol{F}_{\text{int}} = -A^{\mathsf{T}} C A \boldsymbol{u}.}$$

SAY: The compact second-order equation contains connectivity, constitutive physics, and boundary structure. Our job is to learn the compact notation without forgetting that structure.

STOP: If time is nearly over, finish here.

NEXT: Next lecture: replace force balance by a stationary principle. Build a discrete action for an entire path, vary the node values, and watch the discrete Euler–Lagrange equation and endpoint terms appear before taking any continuum limit.

Board checks / quick reference

- Fixed–fixed chain:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}, \quad A^{\mathsf{T}} A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

- Ring:

$$A_{\text{ring}} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix}, \qquad A_{\text{ring}} \mathbf{1} = 0.$$

- Discrete FTC using the full open-chain incidence matrix:

$$\widetilde{A}^{\mathsf{T}} \mathbf{1}_E = -\mathbf{e}^{(0)} + \mathbf{e}^{(N)}, \qquad \mathbf{1}_E^{\mathsf{T}} \widetilde{A} \widetilde{\boldsymbol{u}} = u_N - u_0.$$

- Summation by parts:

$$\boldsymbol{v}^{\mathsf{T}} \widetilde{A} \widetilde{\boldsymbol{u}} = \widetilde{\boldsymbol{u}}^{\mathsf{T}} \widetilde{A}^{\mathsf{T}} \boldsymbol{v}.$$

- Dynamics and force convention:

$$M\ddot{\boldsymbol{u}} + A^{\mathsf{T}} C A \boldsymbol{u} = \boldsymbol{f}, \qquad \boldsymbol{F}_{\text{int}} = -A^{\mathsf{T}} C A \boldsymbol{u}.$$