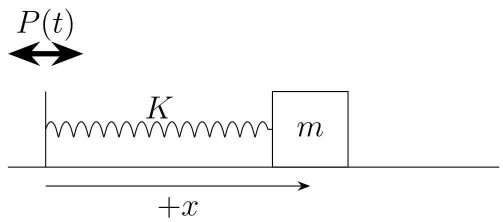
Prof. Mark D Shattuck
Physics 35100 Mechanics
October 31, 2022

Exam 1-retake

Show all of your work!

Question 1. *What is your name?*

Question 2. *Driven Oscillator* A block of mass m is resting on a friction-less table and connected to a movable wall by a linear spring with spring constant K and rest length L so that force is zero if $x-P(t) = L$. It is confined to move in one dimension. Initially the mass is not moving and positioned so the spring is at its rest length. The wall's position P oscillates according to $P(t) = P_0 \sin(\omega t)$.



(1) In terms of the position x, what is the equation of motion.

(2) Consider the mass-spring system:

(a) What is the kinetic, potential, and total energy at $t = 0$?

(b) Is total energy conserved? Why? Why not?

(c) Is momentum conserved? Why? Why not?

(3) Non-dimensionalize the equation using $[L] = L$, $[M] = m$, and $[T] = \frac{1}{\omega_0}$, where $\omega_0 = \sqrt{\frac{K}{m}}$ to obtain an equation of motion of the form:

$$\ddot{u} = -u + A_0 \sin(\Omega\tau),$$

where $\tau = \omega_0 t$ and $u = \frac{x-L}{L}$. Find the values of A_0 and Ω.

(4) Show that:

$$u(\tau) = A\sin(\tau) + B\sin(\Omega\tau)$$

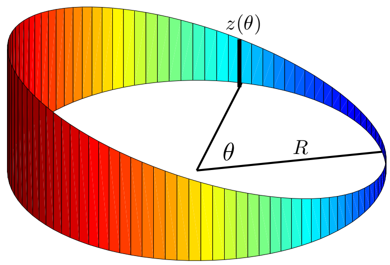
solves the equation:

$$\ddot{u} = -u + A_0\sin(\Omega\tau),$$

Find the values of A and B for the conditions above.

(5) Find the impulse J over 1 period of the drive $0 \leq t \leq 2\pi/\omega$?

Question 3. *Roller coaster* A car of mass M rides on a friction-less three-dimensional track. The track has a circular footprint of radius R such that the x and y coordinates form a circle $x^2 + y^2 = R^2$ in the plane of the ground. The height z of the track varies with the angle θ around the circle $z(\theta) = H(1 - \cos\theta)$.



(1) Find the Lagrangian in terms of θ and $\dot{\theta}$.

(2) Find the generalized momentum p_θ.

(3) Find the equation of motion.

(4) Find the Hamiltonian H as a function of $\dot{\theta}$ and θ. Then express it in terms of p_θ and θ by eliminating $\dot{\theta}$. Is p_θ or H conserved? Why or Why not?

Question 4. *Hamilton's Equations of Motion* The Hamiltonian $H(q_k, p_k, t)$ is a function of K generalized coordinates q_k, K generalized momenta,

$$p_k = \frac{\partial L}{\partial \dot{q}_k},$$

and time t, where L is the Lagrangian. The Lagrangian $L(q_k, \dot{q}_k, t)$ is a function of K generalized coordinates q_k, K generalized velocities $\dot{q}_k$, and time t. The Hamiltonian is related to the Lagrangian by this transformation:

$$H(q_k, p_k, t) = \sum_{k=1}^{K} p_k \dot{q}_k - L(q_k, \dot{q}_k, t).$$

The total differential dF of any function $F(x, .., z)$ is

$$dF = \frac{\partial F}{\partial x} dx + ... + \frac{\partial F}{\partial z} dz.$$

Find the total differential of the transformation equation and use the Euler-Lagrange equations and definition of p_k to show that:

$$\dot{p}_k = -\frac{\partial H}{\partial q_k}, \qquad \dot{q}_k = \frac{\partial H}{\partial p_k}, \qquad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}.$$

These are a set of first order differential equations of motion called Hamilton's equations. (Hint: If differentials like dx, dy, etc are independent so if $Adx + Bdy + ... + Cdz = 0$, then $A = 0$ and $B = 0$, etc.)

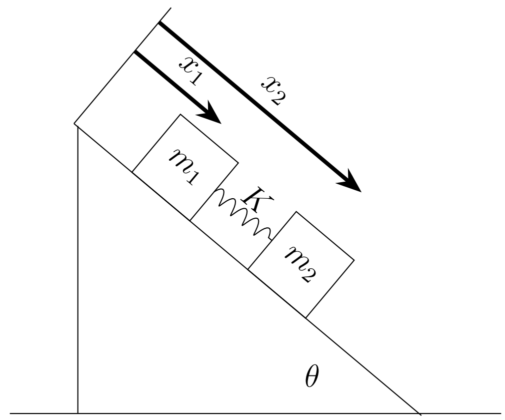
Question 5. *Time dependent Motion.* A particle of mass $m = 3\,\text{kg}$ is initially $t = 0\,s$ at position $\vec{x}(0) = (-2, 0, 3)\,m$. The particle has a velocity

$$\dot{\vec{x}}(t) = (\exp(-t/3), -t, -2\cos t)\,m/s.$$

(1) Find a general expression for the position of the particle $\vec{x}(t)$ and the acceleration $\ddot{x}(t)$ as functions of time t.

(2) What time t is the $\hat{z} = (0, 0, 1)$ component of the acceleration $a_z = \ddot{\vec{x}} \cdot \hat{z}$ *maximum* in the range of $0\,s \leq t \leq 6\,s$? What is the position vector at that point?

Question 6. *Springy blocks on a wedge* Two blocks of mass m_1 and m_2 are connected together by a massless linear spring with spring constant K and sit on a friction-less wedge with angle θ. When $x_2 - x_1 = L$ the spring produces no force. The wedge can not move. The blocks are free to move under the force of gravity pointing downward.



(1) Find the accelerations of the blocks $\ddot{x}_1$ and $\ddot{x}_2$. Note any conserved quantities.

(2) Find the acceleration of the center of mass of the blocks $\ddot{X}_{cm}$ and the acceleration of generalized coordinate that measures the distance between the blocks $\ddot{u} = \ddot{x}_2 - \ddot{x}_1$.