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Physics 35100 Mechanics
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Problem Set 1

Question 1. A 3D Cartesian vector space over the real numbers $\mathbb{R}$ has the following properties:

1) An arbitrary vector $\vec{u} = a\hat{x} + b\hat{y} + c\hat{z}$, where a, b, and c are real numbers, and $\hat{x}$, $\hat{y}$, $\hat{z}$ are basis vectors.
2) $\vec{u} + \vec{v} + \vec{w} = (\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
3) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
4) $\vec{u} + (-\vec{u}) = \vec{0}$
5) $a(b\vec{u}) = (ab)\vec{u}$
6) $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$
7) $(a + b)\vec{u} = a\vec{u} + b\vec{u}$
8) Dot product: $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u} = a$

$$\begin{aligned}
\hat{x} \cdot \hat{x} &= 1, \\
\hat{x} \cdot \hat{y} &= 0, \\
\hat{x} \cdot \hat{z} &= 0, \\
\hat{y} \cdot \hat{x} &= 0, \\
\hat{y} \cdot \hat{y} &= 1, \\
\hat{y} \cdot \hat{z} &= 0, \\
\hat{z} \cdot \hat{x} &= 0, \\
\hat{z} \cdot \hat{y} &= 0, \\
\hat{z} \cdot \hat{z} &= 1,
\end{aligned}$$

Or as a matrix:

$$\begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} \cdot \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \end{bmatrix} = \begin{bmatrix} \hat{x} \cdot \hat{x} & \hat{x} \cdot \hat{y} & \hat{x} \cdot \hat{z} \\ \hat{y} \cdot \hat{x} & \hat{y} \cdot \hat{y} & \hat{y} \cdot \hat{z} \\ \hat{z} \cdot \hat{x} & \hat{z} \cdot \hat{y} & \hat{z} \cdot \hat{z} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbb{I}$$

9) Cross product: $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u} = \vec{w}$

$$\begin{aligned}
\hat{x} \times \hat{x} &= 0, \\
\hat{x} \times \hat{y} &= \hat{z}, \\
\hat{x} \times \hat{z} &= -\hat{y}, \\
\hat{y} \times \hat{x} &= -\hat{z}, \\
\hat{y} \times \hat{y} &= 0, \\
\hat{y} \times \hat{z} &= \hat{x}, \\
\hat{z} \times \hat{x} &= \hat{y}, \\
\hat{z} \times \hat{y} &= -\hat{x}, \\
\hat{z} \times \hat{z} &= 0,
\end{aligned}$$

Or as a matrix:

$$\begin{bmatrix}\hat{x}\\ \hat{y}\\ \hat{z}\end{bmatrix} \times \begin{bmatrix}\hat{x} & \hat{y} & \hat{z}\end{bmatrix} = \begin{bmatrix}\hat{x}\times\hat{x} & \hat{x}\times\hat{y} & \hat{x}\times\hat{z}\\ \hat{y}\times\hat{x} & \hat{y}\times\hat{y} & \hat{y}\times\hat{z}\\ \hat{z}\times\hat{x} & \hat{z}\times\hat{y} & \hat{z}\times\hat{z}\end{bmatrix} = \begin{bmatrix}\hat{0} & \hat{z} & \hat{-y}\\ \hat{-z} & \hat{0} & \hat{x}\\ \hat{y} & \hat{-x} & \hat{0}\end{bmatrix}$$

10) $(\vec{u}+\vec{v})\cdot\vec{w} = \vec{u}\cdot\vec{w}+\vec{v}\cdot\vec{w}$
11) $(\vec{u}+\vec{v})\times\vec{w} = \vec{u}\times\vec{w}+\vec{v}\times\vec{w}$

It is common to describe a vector in a simple form for easy calculatons: Definition: $\vec{u} \equiv (a,b,c) = a\hat{x}+b\hat{y}+c\hat{z}$, where the basis vectors are assumed. Derive the following vector operations in this form and justify each step using the properties above. For example,

1) $\vec{u}+\vec{v} = (a,b,c)+(d,e,f) = ?$

   By definition:

$$(a,b,c)+(d,e,f) = a\hat{x}+b\hat{y}+c\hat{z}+d\hat{x}+e\hat{y}+f\hat{z}$$

   By Property 2 and 3:

$$= (a\hat{x}+d\hat{x})+(b\hat{y}+e\hat{y})+(c\hat{z}+f\hat{z})$$

   By Property 7:

$$= (a+d)\hat{x}+(b+e)\hat{y}+(c+f)\hat{z}$$

   By definition:

$$(a,b,c)+(d,e,f) = (a+d, b+e, c+f)$$

2) $a\vec{u} = a(b,c,d) = ?$
3) $\vec{u}\cdot\vec{v} = (a,b,c)\cdot(d,e,f) = ?$
4) $\vec{u}\times\vec{v} = (a,b,c)\times(d,e,f) = ?$

Question 2. Given $\vec{u} = (1,3,5)$, $\vec{v} = (-3,2,4)$, $\vec{w} = (-1,0,-2)$ using the notation developed above, find:

1) $5\vec{u}+3\vec{v}$
2) $(\vec{u}\cdot\vec{v})\vec{w}$
3) $\vec{u}\times(\vec{v}\times\vec{w})$
4) $(\vec{u}\cdot\vec{w})\vec{v}-(\vec{u}\cdot\vec{v})\vec{w}$

Question 3. If $\vec{u}\cdot\vec{u} = 169$ and $\vec{u}\times\hat{x} = -5\hat{y}$, then what is $\vec{u}$?

Question 4. Any triangle can be represented by three vectors $\vec{u}$, $\vec{v}$, $\vec{w}$.

1) Make a sketch of an example triangle with the tails of $\vec{u}$ and $\vec{v}$ at the same point and $\vec{w}$ pointing from the head of $\vec{v}$ to $\vec{u}$.
2) Find an equation for $\vec{w}$ in terms of $\vec{u}$ and $\vec{v}$.
3) Use the equation to find a relationship between the squared lengths of $\vec{u}$, $\vec{v}$, $\vec{w}$, where the squared length of a vector is $u^2 = |\vec{u}|^2 = \vec{u}\cdot\vec{u}$. Use the relation between the dot product and angle between two vectors to recover the Law of Cosines.
4) Show that the area of the triangle $A = |\vec{u}\times\vec{v}|/2$. Prove the other cross products give the same result $A = |\vec{u}\times\vec{w}|/2 = |\vec{w}\times\vec{v}|/2$.

Question 5. A simple vector space is given by triples (a, b, c) where a, b, c come from the scalar field $\{0, 1, 2\}$, where the addition and multiplication tables for the field are:

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

*	0	1	2
0	0	0	0
1	0	1	2
2	0	2	0

For example, add two vectors $(2, 1, 0) + (1, 2, 2) = (0, 0, 2)$ and multiply a vector by a scalar $2 * (1, 2, 0) = (2, 1, 0)$.

1) List all 27 possible vectors in this space.
2) Show that $\vec{e}_0 = (1, 0, 0)$, $\vec{e}_1 = (2, 1, 0)$, and $\vec{e}_2 = (0, 2, 1)$ are basis vectors that span the space.
3) What is the additive inverse for these vectors? That is find a scalar d such that $(a, b, c) + d * (a, b, c) = (0, 0, 0)$. Often d would be -1, but -1 is not one of the scalar $0, 1, 2$ available.

Question 6. A roller coaster follows the trajectory:

$$\vec{x}(t) = \left(vt, 0, v^2t^2(vt - 10\,m)(vt - 18\,m)/1260\,m^3\right),$$

(using the notation from question 1), for $(-5\,s \le t \le 20\,s)$, and $v = 1\,m/s$.

1) If the height $h(t) = \vec{x}(t) \cdot \hat{z}$, then when (t) and where (x, y, h) does the coaster reach the maximum and minimum heights for $(-5\,s \le t \le 20\,s)$?
2) What is the velocity vector at $t = -5\,s$, the maximum, and minimum heights?
3) What is the acceleration vector at $t = -5\,s$, the maximum and minimum heights?
4) Is the total energy, kinetic plus gravitational, conserved?

Question 7. A 10kg model rocket produces an upward thrust of $49\,N/s^2(3\,s - t)t$ for $3\,s$. At $t = 0\,s$ the rocket is sitting at rest on the launch pad. Assume $g = 9.8\,m/s^2$ and that the mass of the fuel is negligible:

1) Draw a free body diagram of the rocket and pad at $t = 0$?
2) At what time t_{off} will the rocket leave the launch pad?
3) Write down a differential equation for the position of the rocket as a function of time $\vec{x}(t)$ for $0 \le t < t_{off}$.
4) Draw a free body diagram at $t = t_{off}$.
5) Write down a differential equation for $\vec{x}(t)$ for $t_{off} \le t < 3\,s$ and solve for $\vec{x}(t)$. What is the height of the rocket at $t = 3\,s$?
6) What is the velocity at $t = 3\,s$?
7) Use conservation of energy to find the maximum height that rocket reaches. For what times can we use conservation of energy?