From *Classical Mechanics*, R. Douglas Gregory:

**Chapter 4**: 4.14, 4.15 (use Lagrangian Mechanics)

**Question 1. Double Oscillator** Two blocks of mass $m_1 = m$ and $m_2 = \frac{2m}{3}$ are resting at position $x_1$ and $x_2$ on a friction-less table. $m_1$ is connected to a linear spring with spring constant $K_1 = 3K$ and rest length $L_1$. $m_2$ is connected to a second linear spring with spring constant $K_2 = K$ and rest length $L_2$. Both masses are confined to move in one dimension.

![Diagram of double oscillator](image)

1. Write down the Lagrangian in the Cartesian coordinates $x_1$ and $x_2$.
2. Find new generalized coordinates $q_1$ and $q_2$ such that all of the potential terms are of the form $V_k(q_k) = \frac{1}{2}Kq_k^2$. (Hint: $x_1 = q_1 + L_1$ will work for $V_1$.) Then rewrite the Lagrangian in the new coordinates.
3. Find the two conjugate momenta $p_1$ and $p_2$.
4. Find the equations of motion.
5. The equations for the $q_k$ are coupled making them difficult to solve. Show that a change of variables to $\dot{P}_1 = \dot{p}_1 + 2\dot{p}_2$ and $\dot{P}_2 = -2\dot{p}_1 + 3\dot{p}_2$ will uncouple the equations. For example, $\dot{P}_1 = \dot{Q}_1 = -\Omega_1^2 Q_1$, where $Q_1 = 3q_1 + 2q_2$. Show that $Q_1 = 3q_1 + 2q_2$ and obeys $\ddot{Q}_1 = -\Omega_1^2 Q_1$ and find $Q_2$ and $\Omega_1$ and $\Omega_2$ in terms of the base frequency $\sqrt{\frac{K}{m}}$.

**Question 2. Matrix Lagrangian** Find the equation of motion for the Lagrangian:

$$L(\vec{q}, \dot{\vec{q}}) = \frac{1}{2}\vec{q}^T M \vec{q} + \frac{1}{2}\vec{q}^T K \ddot{\vec{q}},$$

where,

$$\vec{q}^T = \begin{bmatrix} \theta & \phi \end{bmatrix},$$

$$M = \begin{bmatrix} 4\mu & 3\mu + \nu \\ 3\mu + \nu & \nu \end{bmatrix},$$

$$K = \begin{bmatrix} 4\kappa & -\kappa \\ -\kappa & \kappa \end{bmatrix}.$$
Question 3. Conservation of Energy Consider a Lagrangian that obeys Euler-Lagrange equations. Show that the Hamiltonian will only be conserved $\dot{H} = 0$ if $L$ does not explicitly depend on time $\frac{\partial L}{\partial t} = 0$. Consider a general Lagrangian $L(q_k(t), \dot{q}_k(t), t)$. (Hint: Find the total time derivative of the Lagrangian along with the Euler-Lagrange equations to relate terms to generalized momenta and their derivatives.)