

Problem Set 4

PHYS V1100: Analytical Dynamics

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Professor Mark Shattuck

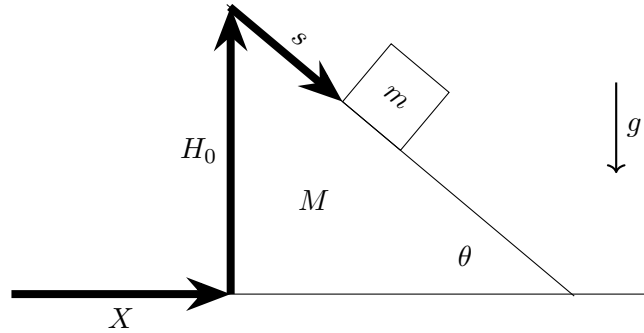
Show enough reasoning that another person can follow your argument. This set emphasizes the transition from generalized coordinates to continuous symmetries and conservation laws, and then to coupled oscillators viewed as a discrete field. Look for structure before doing algebra: identify the symmetry, boundary condition, or local interaction rule that controls the calculation. Computation may be used for checking and visualization, but it should support the mechanical argument rather than replace it. Parts explicitly marked **Optional** or problems labeled **Challenge** are not required for submission. In the oral-presentation problem, all students complete the written portion; only the designated presenter prepares the oral extension.

Problem 1

Block on a movable wedge: cyclic coordinate and coupled acceleration. A block of mass m slides without friction on a wedge of mass M and fixed angle θ . The wedge itself slides without friction on a horizontal table. Gravity acts vertically downward.

Let X be the horizontal displacement of the wedge to the right, and let s measure the displacement of the block *down the incline*. Take the vertical coordinate to increase upward. Up to an irrelevant fixed vertical offset H_0 , the block position in the laboratory frame is

$$\mathbf{r}_m(X, s) = \begin{pmatrix} X + s \cos \theta \\ H_0 - s \sin \theta \end{pmatrix}.$$



- (a) Differentiate the coordinate map and show that the block speed satisfies

$$\dot{\mathbf{r}}_m^2 = \dot{X}^2 + 2\dot{X}\dot{s}\cos\theta + \dot{s}^2.$$

Then write the full Lagrangian in the coordinates (X, s) .

- (b) Identify the cyclic coordinate. Find its conjugate momentum and interpret the conserved quantity physically.
- (c) Derive the two Euler–Lagrange equations and solve them for $\ddot{s}$ and $\ddot{X}$. Show that

$$\ddot{s} = g \frac{(M + m) \sin \theta}{M + m \sin^2 \theta},$$

$$\ddot{X} = -g \frac{m \sin \theta \cos \theta}{M + m \sin^2 \theta}.$$

- (d) Show that the mechanical energy is conserved and write it explicitly. Why does this conservation law coexist with the changing gravitational potential energy of the block?
- (e) Check the limiting cases $M \rightarrow \infty$, $\theta \rightarrow 0$, and $\theta \rightarrow \pi/2$. Interpret each result.

Problem 2

Autonomous Lagrangians and the conserved energy function. Consider a system with generalized coordinates

$$q_1(t), \dots, q_N(t)$$

and Lagrangian

$$L = L(\mathbf{q}, \dot{\mathbf{q}}, t).$$

Define the generalized momenta

$$p_j = \frac{\partial L}{\partial \dot{q}_j}$$

and the Lagrangian energy function

$$E_L = \sum_{j=1}^N p_j \dot{q}_j - L.$$

Assume the motion satisfies the Euler–Lagrange equations.

- (a) Write the total time derivative of the Lagrangian along a trajectory. Be explicit about the distinction between the total derivative dL/dt and the explicit partial derivative $\partial L/\partial t$.
- (b) Differentiate E_L with respect to time and use the Euler–Lagrange equations to show that

$$\frac{dE_L}{dt} = -\frac{\partial L}{\partial t}.$$

- (c) Suppose the Lagrangian has no explicit time dependence,

$$\frac{\partial L}{\partial t} = 0.$$

What follows? Explain why this condition expresses time-translation symmetry of the dynamical description. Does conservation of E_L along one special trajectory by itself prove that L is independent of time everywhere in configuration-velocity space?

- (d) For an ordinary mechanical Lagrangian

$$L = T - V,$$

assume that $V = V(\mathbf{q})$ is velocity independent and that the kinetic energy is quadratic in the generalized velocities,

$$T = \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{M}(\mathbf{q}) \dot{\mathbf{q}},$$

with symmetric $\mathbf{M}$. Show that

$$E_L = T + V.$$

Problem 3

Translation and rotation symmetry of two interacting particles. Two particles of masses m_1 and m_2 move in a plane. Their interaction depends only on their separation,

$$L = \frac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{r}}_2^2 - U(\|\mathbf{r}_2 - \mathbf{r}_1\|).$$

Define

$$\mathbf{p}_i = \frac{\partial L}{\partial \dot{\mathbf{r}}_i} = m_i \dot{\mathbf{r}}_i.$$

- (a) Let $\mathbf{a}$ be an arbitrary constant vector and consider the infinitesimal common translation

$$\delta_{\text{sym}}\mathbf{r}_1 = \epsilon\mathbf{a}, \quad \delta_{\text{sym}}\mathbf{r}_2 = \epsilon\mathbf{a}.$$

Show directly that $\delta_{\text{sym}}L = 0$. Use Noether's theorem to find the conserved quantity associated with this one-parameter transformation.

- (b) Why does the arbitrariness of $\mathbf{a}$ imply conservation of the full vector

$$\boxed{\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2}$$

rather than only one scalar projection of it?

- (c) Introduce the 90° rotation matrix

$$\mathbf{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and consider the infinitesimal common rotation

$$\delta_{\text{sym}}\mathbf{r}_i = \epsilon\mathbf{J}\mathbf{r}_i, \quad i = 1, 2.$$

Show that the kinetic energy and the pair separation are unchanged to first order. Then use Noether's theorem to obtain the conserved scalar angular momentum about the origin.

- (d) Now add a centered isotropic external potential

$$V_{\text{ext}} = \frac{1}{2}\kappa_1\|\mathbf{r}_1\|^2 + \frac{1}{2}\kappa_2\|\mathbf{r}_2\|^2, \quad \kappa_1, \kappa_2 > 0.$$

Which of the two continuous symmetries above survives? Which associated conservation law survives? Explain without deriving the equations of motion.

Problem 4

Written + Oral Presentation

Same local operator, different boundaries: sine and Fourier modes. Consider N identical masses m with nearest-neighbor springs of stiffness κ and lattice spacing a . Away from boundaries, the dimensionless stiffness operator acts as

$$(K\mathbf{q})_n = 2q_n - q_{n-1} - q_{n+1}.$$

The equations of motion are

$$m\ddot{\mathbf{q}} = -\kappa K\mathbf{q}.$$

We will compare two ways of closing the same local nearest-neighbor rule: a fixed-fixed chain and a periodic ring.

- (a) Let A map nodal displacements to signed nearest-neighbor spring extensions, so that

$$\mathbf{e} = A\mathbf{q}, \quad V = \frac{\kappa}{2}\mathbf{e}^T\mathbf{e} = \frac{\kappa}{2}\mathbf{q}^T A^T A\mathbf{q}.$$

For the fixed-fixed chain, take $q_0 = q_{N+1} = 0$ and include the two boundary springs. For the ring, identify q_N periodically with q_0 and include only the N cyclic nearest-neighbor edges. Explain why

$$\boxed{K = A^T A}$$

has the same interior action

$$(K\mathbf{q})_n = 2q_n - q_{n-1} - q_{n+1}$$

in both systems, while the global matrices differ at the boundary. What structural information is carried by A that is not contained in the list of values $\mathbf{q}$ alone?

- (b) Try the local complex pattern

$$q_n \propto e^{ikna}.$$

Show that wherever the nearest-neighbor stencil applies,

$$\boxed{\lambda(k) = 2 - 2\cos(ka) = 4\sin^2\left(\frac{ka}{2}\right)}.$$

Explain why this is a statement about the *local operator*, before any boundary condition has been imposed.

- (c) For fixed endpoints, take moving sites $n = 1, \dots, N$ and impose fictitious endpoint values

$$q_0 = q_{N+1} = 0.$$

Show that the allowed wave numbers are

$$\boxed{k_j a = \frac{j\pi}{N+1}, \quad j = 1, \dots, N,}$$

and that an orthonormal real eigenbasis is

$$\boxed{Q_{nj} = \sqrt{\frac{2}{N+1}} \sin\left(\frac{nj\pi}{N+1}\right)}.$$

Explain why

$$\mathbf{a} = Q^T \mathbf{q}$$

is a discrete sine transform of the site values.

- (d) For the ring, label sites by $n = 0, \dots, N-1$ and impose

$$q_{n+N} = q_n.$$

Show that

$$\boxed{k_r a = \frac{2\pi r}{N}, \quad r = 0, \dots, N-1.}$$

Using the normalized complex basis

$$U_{nr} = \frac{1}{\sqrt{N}} e^{i2\pi rn/N},$$

explain why the modal transform

$$\tilde{\mathbf{q}} = U^\dagger \mathbf{q}$$

is a discrete Fourier transform. Why can a real calculation replace conjugate $+k$ and $-k$ modes by real sine/cosine combinations?

- (e) Compare the lowest mode of the fixed chain with the $r = 0$ mode of the ring. Show that the ring has

$$K\mathbf{1} = 0,$$

while the fixed-fixed chain does not. Explain the difference using continuous symmetry rather than matrix algebra alone.

- (f) For a fixed-fixed chain with $N = 3$, suppose

$$\mathbf{q}(0) = d \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \dot{\mathbf{q}}(0) = 0.$$

Using the normalized sine basis, show that the initial modal amplitudes are

$$\mathbf{a}(0) = d \begin{pmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{pmatrix}.$$

Write the resulting motion as a sum of the three normal modes. You do not need to expand the final answer back into three separate site-coordinate formulas.

- (g) **Structural interpretation / oral extension.** Suppose someone hands you only the column of numbers

$$\mathbf{q} = (q_1, \dots, q_N)^T.$$

Explain why this alone does not tell you whether the values live on a chain, a ring, or part of a higher-dimensional discrete manifold. What extra information is carried by the incidence/interaction structure? In a higher-dimensional model, why might one introduce geometric “virtual edges” or higher simplices that are not additional physical springs?

Problem 5

Challenge

Galilean boosts as a Noether quasi-symmetry. This problem uses the quasi-symmetry form of Noether’s theorem. If an infinitesimal transformation

$$\delta_{\text{sym}} q = \epsilon h$$

changes the Lagrangian by a total derivative,

$$\delta_{\text{sym}} L = \epsilon \frac{dB}{dt},$$

then along an Euler–Lagrange trajectory

$$\boxed{ph - B = \text{constant.}}$$

- (a) For a free particle in one dimension,

$$L = \frac{1}{2}m\dot{x}^2,$$

consider the infinitesimal Galilean boost

$$x \longrightarrow x' = x + \epsilon t.$$

Find $\delta_{\text{sym}}x$ and $\delta_{\text{sym}}\dot{x}$. Show, to first order in ϵ , that

$$\delta_{\text{sym}}L = \epsilon \frac{d}{dt}(mx).$$

- (b) Identify h and B . Use the quasi-symmetry form of Noether’s theorem to show that

$$\boxed{tp - mx = \text{constant.}}$$

Verify the result directly using the free-particle equation of motion.

- (c) Interpret this conservation law. Show that it is equivalent to uniform straight-line motion

$$x(t) = x_0 + vt.$$

Why is this conservation law not the same statement as conservation of momentum, even though both hold for the free particle?

- (d) **Optional extension.** For N particles in one dimension with

$$L = \sum_{i=1}^N \frac{1}{2}m_i\dot{x}_i^2 - V(\{x_i - x_j\}),$$

apply the common boost

$$\delta_{\text{sym}}x_i = \epsilon t.$$

Show that the corresponding Noether charge can be written

$$\boxed{tP - MX_{\text{cm}} = \text{constant},}$$

where

$$P = \sum_i m_i \dot{x}_i, \quad M = \sum_i m_i, \quad X_{\text{cm}} = \frac{1}{M} \sum_i m_i x_i.$$

What physical statement does this encode?

Additional practice, not required. The following bank problem works through the exact sine spectrum of a fixed-boundary chain and includes an optional numerical check and continuum-limit interpretation.

Problem 6

Optional Practice

Fixed-boundary chain: sine modes and the discrete spectrum. Consider N identical point masses m moving along a line between two fixed walls. Consecutive nodes are joined by identical springs of stiffness k . Let $u_j(t)$ be the displacement of mass j , with fixed endpoint values

$$u_0(t) = u_{N+1}(t) = 0.$$

For the N free masses, write

$$\mathbf{u} = (u_1, \dots, u_N)^T.$$

For unit spring stiffness, the fixed-boundary stiffness matrix is

$$\mathbf{K}_N = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix}.$$

The equations of motion are therefore

$$\boxed{m\ddot{\mathbf{u}} + k\mathbf{K}_N\mathbf{u} = 0.}$$

(a) For

$$r = 1, \dots, N, \quad \theta_r = \frac{r\pi}{N+1},$$

consider the vector $\mathbf{y}^{(r)}$ with components

$$\boxed{y_j^{(r)} = \sin(j\theta_r), \quad j = 1, \dots, N.}$$

Explain how the fixed endpoint conditions are already encoded by this choice.

(b) Apply an interior row of $\mathbf{K}_N$ to $\mathbf{y}^{(r)}$. Using

$$\sin((j+1)\theta) + \sin((j-1)\theta) = 2\sin(j\theta)\cos\theta,$$

show that

$$\boxed{\mathbf{K}_N\mathbf{y}^{(r)} = \lambda_r\mathbf{y}^{(r)}}, \quad \boxed{\lambda_r = 2(1 - \cos\theta_r) = 4\sin^2\left(\frac{\theta_r}{2}\right)}.$$

(c) Seek a normal-mode motion

$$\mathbf{u}(t) = a_r(t)\mathbf{y}^{(r)}.$$

Show that a_r is a simple harmonic oscillator and derive the normal frequency

$$\boxed{\omega_r = 2\sqrt{\frac{k}{m}}\sin\left(\frac{r\pi}{2(N+1)}\right)}.$$

What is the physical meaning of increasing r ?

- (d) For $N = 5$, calculate the five analytic eigenvalues λ_r . Use MATLAB or Python to compute the eigenvalues and eigenvectors of $\mathbf{K}_5$, and compare them with the analytic formulas. Plot the five mode shapes $y_j^{(r)}$ against the node index j .
- (e) **Optional continuum-limit interpretation.** Let the node spacing be a and the total fixed-end length be

$$\ell = (N + 1)a.$$

Show that for low mode number $r \ll N$,

$$\omega_r \approx \sqrt{\frac{k}{m}} \frac{r\pi a}{\ell}.$$

If the discrete chain approximates a string with mass density ρ and tension T , so that $m = \rho a$ and $k = T/a$, recover the fixed-end string spectrum.