

Problem Set 3

PHYS V1100: Analytical Dynamics

Fall 2026

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Show enough reasoning that another person can follow your argument. The required work emphasizes variational structure, generalized coordinates, and constrained mechanics. Parts explicitly marked **Optional**, **Optional extension**, or **Optional challenge** are not required for submission, but they are included because they develop useful consequences of the main ideas. Computation should be used for investigation or validation rather than only to generate an answer. In the oral-presentation problem, all students complete the written portion; only the designated presenter prepares the oral extension.

Problem 1

From a discrete action to a discrete equation of motion. Consider a one-dimensional discrete path

$$x_0, x_1, \dots, x_N$$

with equally spaced times separated by Δt . Let the discrete action be

$$S_d = \sum_{n=0}^{N-1} L_d(x_n, x_{n+1}),$$

with

$$L_d(x_n, x_{n+1}) = \frac{m}{2\Delta t}(x_{n+1} - x_n)^2 - \Delta t V(x_n).$$

The endpoints are fixed. For an interior point x_n , you may use the discrete Euler–Lagrange equation

$$D_2 L_d(x_{n-1}, x_n) + D_1 L_d(x_n, x_{n+1}) = 0.$$

Here D_1 and D_2 mean differentiation with respect to the first and second arguments of L_d .

(a) Compute

$$D_2 L_d(x_{n-1}, x_n) \quad \text{and} \quad D_1 L_d(x_n, x_{n+1}),$$

and show that the discrete Euler–Lagrange equation becomes

$$m \frac{x_{n+1} - 2x_n + x_{n-1}}{\Delta t^2} = -V'(x_n).$$

(b) The equation above resembles the continuum equation

$$m\ddot{x} = -V'(x).$$

State carefully which of the following is exact and which involves an approximation:

(i) the difference equation derived from S_d ;

(ii) interpreting

$$\frac{x_{n+1} - 2x_n + x_{n-1}}{\Delta t^2}$$

as an approximation to $\ddot{x}(t_n)$ for a smooth continuum trajectory.

Why is this distinction important?

(c) Introduce the step variable

$$s_n = x_{n+1} - x_n.$$

Rewrite the second-order difference equation as a pair of first-order update statements. Arrange them so that they match the move-then-turn convention used in Lecture 4:

move to the new position first, then change the step using the force there.

(d) **Optional extension: nonlinear potential.** Now take

$$V(x) = \frac{1}{2}kx^2 + \frac{\alpha}{4}x^4, \quad k > 0,$$

with α not necessarily small. Write the exact discrete equation of motion and the corresponding move-then-turn update. Which term makes the update nonlinear?

(e) For a free particle, $V'(x) = 0$. What does the discrete Euler–Lagrange equation say about s_n ? Relate the result to generalized momentum for this discrete mechanical action.

Problem 2

Stationary action need not be a minimum. Consider the harmonic oscillator

$$L(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2$$

on the time interval $0 \leq t \leq T$, with fixed endpoint conditions

$$x(0) = x(T) = 0.$$

The path

$$x_{\text{cl}}(t) = 0$$

is one classical trajectory satisfying these endpoints. We will probe the action near this path using neighboring paths that satisfy the same endpoint conditions.

(a) Verify directly from the Euler–Lagrange equation that $x_{\text{cl}}(t) = 0$ is a classical trajectory.

(b) Consider the one-parameter family of neighboring paths

$$x_\epsilon(t) = \epsilon \sin\left(\frac{\pi t}{T}\right).$$

Explain why every member of this family has the same endpoint values as x_{cl} . Compute $\dot{x}_\epsilon$ and evaluate the action

$$S(\epsilon) = \int_0^T L(x_\epsilon, \dot{x}_\epsilon) dt.$$

Show that

$$S(\epsilon) - S(0) = \frac{m\epsilon^2 T}{4} \left[\left(\frac{\pi}{T}\right)^2 - \omega^2 \right].$$

(c) Show explicitly that

$$\left. \frac{dS}{d\epsilon} \right|_{\epsilon=0} = 0$$

for every value of T . Then determine whether $S(\epsilon)$ bends upward, is flat, or bends downward at $\epsilon = 0$ in each of the cases

$$T < \frac{\pi}{\omega}, \quad T = \frac{\pi}{\omega}, \quad T > \frac{\pi}{\omega}.$$

What does this show about the statement “the physical path minimizes the action”?

(d) **Computational validation.** Set $m = 1$ and $\omega = 1$. Use numerical quadrature applied directly to

$$S[x] = \int_0^T \left(\frac{1}{2} \dot{x}^2 - \frac{1}{2} x^2 \right) dt$$

for the paths

$$x_\epsilon(t) = \epsilon \sin \left(\frac{\pi t}{T} \right).$$

Do this for at least two choices of T , one below π and one above π , and for several positive and negative values of ϵ near zero.

Plot or tabulate S versus ϵ . Compare the numerical result with the analytic expression from part (b), and explain what feature of the plot verifies stationarity at $\epsilon = 0$ and what feature distinguishes a local minimum along this family from a local maximum along this family.

(e) **Optional challenge: higher variation modes.** Replace the trial family by

$$x_{\epsilon,n}(t) = \epsilon \sin \left(\frac{n\pi t}{T} \right), \quad n = 1, 2, 3, \dots$$

Show that

$$S_n(\epsilon) - S_n(0) = \frac{m\epsilon^2 T}{4} \left[\left(\frac{n\pi}{T} \right)^2 - \omega^2 \right].$$

For fixed T , determine which mode numbers give positive, zero, or negative curvature of the action at the classical path. Explain why, once $T > \pi/\omega$, the path $x = 0$ can be stationary without being a minimum in the full space of paths.

What is special when

$$T = \frac{n\pi}{\omega}$$

for some integer n ?

Problem 3

Written + Oral Presentation

From a global action to a local equation. A central feature of variational mechanics is that one scalar quantity associated with an entire path produces an equation that must hold locally at every interior point. The written portion is for everyone. The oral extension is prepared only by the student assigned to present this problem in class.

Written portion — all students submit.

For a discrete path with fixed endpoint variations, suppose the first variation has been reduced to

$$\delta S_d = \sum_{n=1}^{N-1} E_n \delta q_n.$$

For a continuum path with fixed endpoint variations, suppose integration by parts has reduced the first variation to

$$\delta S = \int_{t_1}^{t_2} E(t) \eta(t) dt, \quad \eta(t_1) = \eta(t_2) = 0.$$

Here E_n and $E(t)$ denote the corresponding discrete and continuum Euler–Lagrange expressions.

- (a) **Discrete localization.** Choose a variation for which every δq_n vanishes except at one interior node k . Use stationarity to show that $E_k = 0$. Why does repeating this argument for every interior node turn one global stationarity statement into a local difference equation along the whole path?
- (b) **Continuum localization.** Suppose $E(t)$ is continuous and assume, for contradiction, that $E(t_*) > 0$ at some interior time t_* . Explain how to choose an allowed variation $\eta(t)$ localized near t_* so that

$$\int_{t_1}^{t_2} E(t) \eta(t) dt > 0.$$

Why does this contradict stationarity, and what conclusion follows about $E(t)$?

- (c) **The same structure in two languages.** Complete the continuum side of the following correspondence and give one sentence describing the common role of each pair.

discrete path	continuum path
δq_n at an interior node	?
difference / summation-by-parts structure	?
δq_n arbitrary independently	?
endpoint nodes and boundary contributions	?

- (d) **Why one allowed variation is not enough.** Suppose that, instead of allowing arbitrary fixed-endpoint variations, we allowed only variations of one prescribed shape,

$$\eta(t) = \epsilon \phi(t), \quad \phi(t_1) = \phi(t_2) = 0.$$

Stationarity would then imply only

$$\int_{t_1}^{t_2} E(t) \phi(t) dt = 0.$$

Explain why this single condition does *not* imply $E(t) = 0$ pointwise. What freedom has been lost compared with Hamilton’s principle as used in class?

- (e) **What does “vary the action” actually mean?** Why is it incomplete to say that the Euler–Lagrange equation comes from “differentiating the action”? Give a short explanation that includes the object being varied, the role of integration or summation by parts, and the role of arbitrary interior variations.

Oral extension — designated presenter only. Prepare a short board presentation organized around the question

How can a global action produce a local equation of motion?

Your presentation should do all of the following:

- (i) Show one discrete equation and one continuum equation that display the common structure.
- (ii) Explain where the boundary term comes from and why the fixed-endpoint condition matters.
- (iii) Explain why arbitrary *localizable* variations, rather than one prescribed global variation, are what force the Euler–Lagrange expression to vanish pointwise.
- (iv) Give one explicit example showing that orthogonality to a single allowed variation does not force a function to vanish. For example, on $0 \leq t \leq T$ compare

$$\phi(t) = \sin\left(\frac{\pi t}{T}\right)$$

with a nonzero function that has zero inner product with ϕ .

- (v) End with a one-sentence statement of the common chain

variation $\longrightarrow$ move the difference/derivative $\longrightarrow$ boundary + interior $\longrightarrow$ local equation.

Problem 4

Test your understanding: variations, boundaries, and constraints. The parts below are intended to be short. In most cases the important structure is already provided; supply the missing step and explain what it means physically or mathematically.

- (a) **A global discrete action gives a local equation.** Consider

$$\mathcal{S}_d = \sum_j L\left(x_j, \frac{x_{j+1} - x_j}{\Delta t}\right) \Delta t.$$

An interior node x_n appears only in the terms labeled $n - 1$ and n . Define

$$\dot{x}_n = \frac{x_{n+1} - x_n}{\Delta t}, \quad \dot{x}_{n-1} = \frac{x_n - x_{n-1}}{\Delta t}.$$

Compute

$$\frac{\partial \dot{x}_n}{\partial x_n}, \quad \frac{\partial \dot{x}_{n-1}}{\partial x_n}.$$

Why are the opposite signs the essential ingredient that turns the variation into a difference of neighboring generalized momenta?

- (b) **What kills the continuum boundary term?** After integrating by parts, the first variation of the action may be written

$$\delta S = \left[\frac{\partial L}{\partial \dot{q}} \delta q \right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \right] \delta q dt.$$

- (i) What assumption makes the boundary term vanish in the derivation used in class?
- (ii) Suppose $q(t_1)$ is fixed but $q(t_2)$ is allowed to vary. Which term can no longer simply be discarded?
- (iii) Why is it misleading to describe boundary terms as “unimportant terms that vanish”?

(c) **Same equations, different Lagrangian.** Suppose

$$L'(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{dF(q, t)}{dt}.$$

Complete

$$S' - S = \int_{t_1}^{t_2} \frac{dF}{dt} dt = \underline{\hspace{4cm}}.$$

Why does this not change the Euler-Lagrange equations for fixed endpoint data?

- (d) **Actual displacement versus virtual displacement.** A bead is constrained to a frictionless circular hoop of radius R whose center translates horizontally according to a prescribed function $a(t)$. Use the generalized coordinate θ :

$$\mathbf{r}(\theta, t) = \begin{pmatrix} a(t) + R \cos \theta \\ R \sin \theta \end{pmatrix}.$$

The actual differential is

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial \theta} d\theta + \frac{\partial \mathbf{r}}{\partial t} dt.$$

Write the virtual displacement $\delta \mathbf{r}$. Which displacement contains the motion of the hoop itself, and why?

- (e) **Generalized force is a projection.** For an applied force $\mathbf{F}$, write

$$\delta W = \mathbf{F} \cdot \delta \mathbf{r} = Q_\theta \delta \theta.$$

Using the result above, show in one step that

$$Q_\theta = \mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial \theta}.$$

Why need Q_θ not have the dimensions of an ordinary force?

- (f) **“No work” versus “no virtual work.”** Let $\mathbf{R}$ be the normal reaction force exerted by the frictionless hoop. For an ideal constraint, which statement is guaranteed?
- (i) $\mathbf{R} = 0$,
 - (ii) $\mathbf{R} \cdot d\mathbf{r} = 0$,
 - (iii) $\mathbf{R} \cdot \delta \mathbf{r} = 0$.

For the translating hoop, explain briefly why statement (iii) can be true even when statement (ii) is not.

Problem 5

Two particles of masses m_1 and m_2 move in a plane and interact through an internal potential that depends only on their separation. Define

$$M = m_1 + m_2, \quad \mu = \frac{m_1 m_2}{M},$$

and introduce the center-of-mass and relative coordinates

$$\mathbf{X} = \frac{m_1 \mathbf{x}_1 + m_2 \mathbf{x}_2}{M}, \quad \mathbf{l} = \mathbf{x}_2 - \mathbf{x}_1.$$

The inverse transformation is

$$\mathbf{x}_1 = \mathbf{X} - \frac{m_2}{M} \mathbf{l}, \quad \mathbf{x}_2 = \mathbf{X} + \frac{m_1}{M} \mathbf{l}.$$

For the spring interaction considered previously,

$$V(l) = \frac{1}{2} K (l - l_0)^2, \quad l = \|\mathbf{l}\|,$$

but several parts below hold for any potential $V(l)$.

- (a) Differentiate the inverse transformation to express $\dot{\mathbf{x}}_1$ and $\dot{\mathbf{x}}_2$ in terms of $\dot{\mathbf{X}}$ and $\dot{\mathbf{l}}$.
- (b) The kinetic energy in the original particle coordinates is

$$T = \frac{1}{2} m_1 \dot{\mathbf{x}}_1^2 + \frac{1}{2} m_2 \dot{\mathbf{x}}_2^2.$$

Substitute the expressions above and show that

$$\boxed{T = \frac{1}{2} M \dot{\mathbf{X}}^2 + \frac{1}{2} \mu \dot{\mathbf{l}}^2}.$$

In particular, identify why the mixed term $\dot{\mathbf{X}} \cdot \dot{\mathbf{l}}$ cancels.

- (c) Show that the total linear momentum is

$$\mathbf{P} = m_1 \dot{\mathbf{x}}_1 + m_2 \dot{\mathbf{x}}_2 = M \dot{\mathbf{X}}.$$

For an isolated system, what does this say about the $\mathbf{X}$ part of the motion?

- (d) Rewrite the total mechanical energy in the new coordinates. Explain precisely what has separated.
- (e) Derive the equation of motion for the relative coordinate directly from Newton's equations for the two particles. Show that for the spring

$$\boxed{\mu \ddot{\mathbf{l}} = -K(l - l_0) \hat{\mathbf{l}}}.$$

What is the corresponding result for a general central potential $V(l)$?

- (f) **Optional interpretation: reduced-mass limits.** Interpret the reduced mass by considering two limits.
 - (i) If $m_2 \gg m_1$, what does μ approach, and what familiar physical picture emerges?

- (ii) If $m_1 = m_2 = m$, find M and μ . Why is the effective mass for the relative motion smaller than either individual mass?
- (g) Summarize the structural simplification achieved by the coordinate change. What has happened to a two-particle problem?

Problem 6

Bead on a parabola: generalized coordinates and generalized force. A bead of mass m slides without friction on a fixed wire in a vertical plane. The wire is the parabola

$$y = ax^2, \quad a > 0,$$

with y measured upward. Use x itself as the generalized coordinate, so that the coordinate map is

$$\mathbf{r}(x) = \begin{pmatrix} x \\ ax^2 \end{pmatrix}.$$

Gravity is the only applied force other than the normal reaction of the wire:

$$\mathbf{F}_g = \begin{pmatrix} 0 \\ -mg \end{pmatrix}.$$

- (a) Differentiate the coordinate map to find

$$\frac{\partial \mathbf{r}}{\partial x}, \quad \dot{\mathbf{r}},$$

and hence show that the kinetic energy is

$$T = \frac{1}{2}m(1 + 4a^2x^2)\dot{x}^2.$$

What does the factor $1 + 4a^2x^2$ tell you geometrically about using x as the coordinate along the wire?

- (b) Compute the generalized force conjugate to x directly from virtual work,

$$Q_x = \mathbf{F}_g \cdot \frac{\partial \mathbf{r}}{\partial x}.$$

Show that

$$Q_x = -2mga x.$$

Explain briefly why the normal reaction force from the wire does not appear in Q_x .

- (c) Use Lagrange's equation in generalized-force form,

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{x}} - \frac{\partial T}{\partial x} = Q_x,$$

to derive the exact equation of motion

$$(1 + 4a^2x^2)\ddot{x} + 4a^2x\dot{x}^2 + 2agx = 0.$$

Do not skip the derivative of the position-dependent coefficient in T .

- (d) Now solve the same problem using a potential. Take

$$V(x) = mgy = mgax^2$$

and construct

$$L = T - V.$$

Apply the Euler–Lagrange equation and verify that you recover the same equation of motion as in part (c). What structural equivalence are you checking by doing the problem both ways?

- (e) **Optional extension: small-amplitude motion.** The bottom of the parabola is at $x = 0$. Consider small-amplitude motion near the bottom. Treat x and $\dot{x}$ as small quantities of the same order in the oscillation amplitude and keep only terms linear in the amplitude. Show that the exact nonlinear equation reduces to

$$\ddot{x} + 2agx = 0.$$

Hence identify the small-amplitude angular frequency. Which equation is exact, and which equation is an approximation?

- (f) **Optional extension: energy check.** The mechanical energy is

$$E = T + V = \frac{1}{2}m(1 + 4a^2x^2)\dot{x}^2 + mgax^2.$$

Use the exact equation of motion to verify directly that

$$\frac{dE}{dt} = 0.$$

Why is this a useful check on the algebra above?