

Problem Set 2

PHYS V1100: Analytical Dynamics

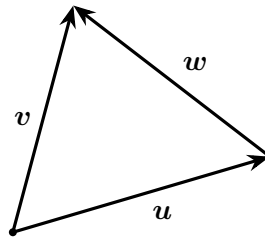
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Show enough reasoning that another person can follow your argument. Computation may be used where requested, but the purpose is to expose the mechanical structure, not just to obtain a numerical answer. If vector or matrix differentiation feels rusty, see the last question.

Problem 1

Let $\mathbf{u}$ and $\mathbf{v}$ be two nonparallel vectors with a common tail, and let $\mathbf{w}$ point from the head of $\mathbf{u}$ to the head of $\mathbf{v}$, as shown schematically below.



This problem deliberately builds the geometry in stages. At each stage, pay attention to what additional mathematical structure is being used.

- (a) Use only vector addition and scalar multiplication to express $\mathbf{w}$ in terms of $\mathbf{u}$ and $\mathbf{v}$. What mathematical structure is required for this statement?
- (b) Now suppose the vector space is equipped with an inner product. Define

$$u^2 = \mathbf{u} \cdot \mathbf{u}, \quad v^2 = \mathbf{v} \cdot \mathbf{v}, \quad w^2 = \mathbf{w} \cdot \mathbf{w}.$$

Use part (a) to derive a relation among u^2 , v^2 , w^2 , and $\mathbf{u} \cdot \mathbf{v}$. Then use

$$\mathbf{u} \cdot \mathbf{v} = uv \cos \theta$$

to recover the Law of Cosines, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$. What new structure has been introduced compared with part (a)?

- (c) Now specialize to three-dimensional Euclidean space. Show that the area $\mathcal{A}$ of the triangle is

$$\mathcal{A} = \frac{1}{2} |\mathbf{u} \times \mathbf{v}|.$$

Then use $\mathbf{w} = \mathbf{v} - \mathbf{u}$ to prove algebraically that

$$\mathbf{u} \times \mathbf{w} = \mathbf{u} \times \mathbf{v}, \quad \mathbf{w} \times \mathbf{v} = -\mathbf{u} \times \mathbf{v}.$$

Explain why all three cross products give the same (unsigned) triangle area even though one of them has the opposite orientation.

- (d) A more geometric way to represent an oriented plane element is with the wedge product $\mathbf{u} \wedge \mathbf{v}$. Use $\mathbf{w} = \mathbf{v} - \mathbf{u}$ to show

$$\mathbf{u} \wedge \mathbf{w} = \mathbf{u} \wedge \mathbf{v}, \quad \mathbf{w} \wedge \mathbf{v} = -\mathbf{u} \wedge \mathbf{v}.$$

What geometric information does the bivector $\mathbf{u} \wedge \mathbf{v}$ represent? Why might it be regarded as more fundamental than the cross product for describing oriented area?

Problem 2

A roller coaster of mass M is prescribed to follow the trajectory

$$\mathbf{x}(t) = (1 \text{ m}) \left[\tau \hat{\mathbf{x}} + \frac{\tau^2(\tau - 10)(\tau - 18)}{1260} \hat{\mathbf{z}} \right], \quad \tau \equiv \frac{t}{1 \text{ s}},$$

for $-5 \leq \tau \leq 20$. The x -axis is horizontal and the z -axis points upward. Take the gravitational acceleration to be

$$g_0 = 9.8 \text{ m/s}^2.$$

The word *prescribed* is important: the function $\mathbf{x}(t)$ gives a particular history of the system. In this problem you will ask what can, and what cannot, be inferred about the dynamics that produced that history.

Define

$$f(\tau) = \frac{\tau^2(\tau - 10)(\tau - 18)}{1260},$$

so that the height is $h(t) = (1 \text{ m})f(\tau)$.

- Is the specified function $\mathbf{x}(t)$ itself a dynamical law? What does it determine about the net force along this particular history, and what does it *not* determine about the force law for arbitrary states or initial conditions?
- Find the velocity $\mathbf{v}(t)$ and acceleration $\mathbf{a}(t)$. Write the corresponding Newtonian state $S(t)$.
- Find all stationary points of the height $h(t)$ in the interior of the time interval and classify them as local maxima or minima. Then determine the global maximum and minimum heights on the full interval $-5 \leq \tau \leq 20$. Be careful about the endpoints.
- At each interior stationary height, is the coaster instantaneously at rest? Must its acceleration vanish there? What is different about the global maximum at $t = -5 \text{ s}$?
- Define the mechanical energy associated with kinetic energy and gravitational potential energy by

$$E(t) = \frac{1}{2}M\mathbf{v}^2(t) + Mg_0h(t).$$

Is E conserved along the prescribed trajectory? Give a simple demonstration without needing to analyze the entire function $E(t)$.

- Suppose the Newtonian equation of motion is written as

$$M\mathbf{a} = -Mg_0\hat{\mathbf{z}} + \mathbf{F}_{\text{other}},$$

where $\mathbf{F}_{\text{other}}$ represents everything other than gravity. Show that

$$\frac{dE}{dt} = \mathbf{F}_{\text{other}} \cdot \mathbf{v}.$$

For the prescribed trajectory, express dE/dt in terms of h , and verify at $t = 3\text{ s}$ that it is nonzero. Interpret the result physically.

- (g) Could gravity plus the normal force from an ideal, fixed, frictionless track produce this prescribed motion by themselves? Explain. What kinds of additional physics could account for the prescribed trajectory?

Problem 3

Two particles of masses m_1 and m_2 move in the x - y plane. Their positions are $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$. They interact through a spring whose potential energy depends only on the distance between the particles,

$$V(l) = \frac{1}{2}K(l - l_0)^2,$$

where

$$\mathbf{l} = \mathbf{x}_2 - \mathbf{x}_1, \quad l = \|\mathbf{l}\|, \quad \hat{\mathbf{l}} = \frac{\mathbf{l}}{l}.$$

Assume $l \neq 0$ throughout the problem and that there are no external forces.

- (a) Consider a common translation of both particles,

$$\mathbf{x}_1 \mapsto \mathbf{x}_1 + \mathbf{a}, \quad \mathbf{x}_2 \mapsto \mathbf{x}_2 + \mathbf{a},$$

where $\mathbf{a}$ is constant. Show that $\mathbf{l}$, l , and V are unchanged. What physical feature of the interaction does this make explicit?

- (b) Using

$$\nabla_{\mathbf{x}_1} l = -\hat{\mathbf{l}}, \quad \nabla_{\mathbf{x}_2} l = +\hat{\mathbf{l}},$$

(which you may derive directly or take from the supplemental background problem), find the force on each particle from

$$\mathbf{F}_i = -\nabla_{\mathbf{x}_i} V.$$

Check the directions of the forces for both a stretched spring ($l > l_0$) and a compressed spring ($l < l_0$).

- (c) Analyze Newton's third law for this interaction.

- (i) Show that the two forces obey the equal-and-opposite form of the third law.
- (ii) Show that the forces also obey the strong form of the third law: they act along the line joining the particles. Use this to show that the total internal torque about any origin vanishes.
- (iii) Which assumption is responsible for the equal-and-opposite property, and which stronger assumption is responsible for the line-of-centers property?

- (d) Define the total mass and center-of-mass position by

$$M = m_1 + m_2, \quad \mathbf{X} = \frac{m_1 \mathbf{x}_1 + m_2 \mathbf{x}_2}{M}.$$

Use Newton's second law and the result above to find $\ddot{\mathbf{X}}$. What does the result say about the motion of the center of mass?

- (e) Replace the two particle coordinates $(\mathbf{x}_1, \mathbf{x}_2)$ by the center-of-mass and relative coordinates

$$\mathbf{X} = \frac{m_1 \mathbf{x}_1 + m_2 \mathbf{x}_2}{M}, \quad \mathbf{l} = \mathbf{x}_2 - \mathbf{x}_1.$$

Find the inverse transformation: express $\mathbf{x}_1$ and $\mathbf{x}_2$ in terms of $\mathbf{X}$ and $\mathbf{l}$.

- (f) Rewrite the potential V in the new coordinates $(\mathbf{X}, \mathbf{l})$. Why is the absence of $\mathbf{X}$ important? Explain, without doing the full kinetic-energy calculation yet, why these coordinates are promising for separating the overall motion of the system from its internal motion.

Problem 4

Oral Presentation

Let a particle of constant mass m move in the plane with position $\mathbf{x}(t)$, velocity $\mathbf{v} = \dot{\mathbf{x}}$, and force $\mathbf{F}(\mathbf{x})$. In this problem we will treat potential energy as something to be *discovered* from the work done by the force.

- (a) Starting from Newton's law

$$m\dot{\mathbf{v}} = \mathbf{F},$$

take the dot product with $\mathbf{v}$ and show that

$$\frac{d}{dt} \left(\frac{1}{2} m \mathbf{v} \cdot \mathbf{v} \right) = \mathbf{F} \cdot \mathbf{v}.$$

Explain why the right-hand side is naturally interpreted as instantaneous power.

- (b) Along an infinitesimal displacement $d\mathbf{x}$ the work done by the force is

$$\delta W = \mathbf{F} \cdot d\mathbf{x}.$$

Suppose there exists a scalar function $V(\mathbf{x})$ for which

$$\mathbf{F} \cdot d\mathbf{x} = -dV$$

for every infinitesimal displacement. Use

$$dV = (\nabla V)^T d\mathbf{x}$$

to show that

$$\mathbf{F} = -\nabla V.$$

Then show directly that $E = T + V$ is conserved.

- (c) Consider the force field

$$\mathbf{F}(x, y) = -(2x + y) \hat{\mathbf{x}} - (x + 2y) \hat{\mathbf{y}}.$$

Find a potential $V(x, y)$ such that $\mathbf{F} = -\nabla V$. Check that the mixed partial-derivative condition expected for a conservative field is satisfied.

(d) Now consider

$$\mathbf{G}(x, y) = -y \hat{\mathbf{x}} + x \hat{\mathbf{y}}.$$

Show that it cannot be written globally as $-\nabla V$ in the plane. Then calculate the work around the unit circle, oriented counterclockwise, using

$$x = \cos \theta, \quad y = \sin \theta, \quad 0 \leq \theta \leq 2\pi.$$

How does this result distinguish $\mathbf{G}$ from the field in part (c)?

(e) Summarize the logical chain connecting Newton's law, exact work, and energy conservation. In particular, distinguish the statements “ $\mathbf{F} \cdot d\mathbf{x}$ is exact” and “ $\mathbf{F}$ does no work.”

Problem 5

Consider an ordered sequence of configurations $x_0, x_1, x_2, \dots$ in one dimension. Before assigning a physical clock, define the *step*

$$s_n = x_{n+1} - x_n$$

and the *turn*

$$\kappa_n = s_n - s_{n-1}, \quad n \geq 1.$$

The word “turn” means any change in the step: it may change its magnitude, its direction, or both.

(a) Show that

$$s_n = s_0 + \sum_{j=1}^n \kappa_j$$

and then show that

$$x_n = x_0 + ns_0 + \sum_{j=1}^{n-1} (n-j)\kappa_j.$$

Write the second expression first as a double sum and then reduce it to the single weighted sum. Explain the analogy with integrating acceleration once to obtain velocity and twice to obtain position.

(b) In more than one dimension, decompose a turn κ into a component parallel to the current step $\mathbf{s}$ and a component perpendicular to it. What qualitative effect does each component have? What special cases correspond to a turn angle of 0 or 180°?

(c) Let the state be $S_n = (x_n, s_n)$ and consider the semi-explicit update

$$\begin{aligned} x_{n+1} &= x_n + s_n, \\ s_{n+1} &= s_n + K(x_{n+1}, s_n, t_n). \end{aligned}$$

For $K = 0$, derive x_n . For constant $K = K_0$, derive both s_n and x_n .

(d) Now assign equally spaced clock times $t_n = n\Delta$. Define

$$v_n = \frac{s_n}{\Delta}, \quad a_n \approx \frac{\kappa_n}{\Delta^2}.$$

Show that Newton's law gives the discrete turn rule

$$\kappa \approx \Delta^2 M^{-1} F.$$

For one particle, apply this result to:

- (i) uniform gravity, $F = mg$;
- (ii) a spring, $F = -kx$.

In the spring case identify the dimensionless parameter c in $K = -cx$ and express it in terms of $\omega = \sqrt{k/m}$ and Δ .

- (e) For the spring-like rule $K(x_{n+1}) = -cx_{n+1}$, show that

$$\begin{pmatrix} x_{n+1} \\ s_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -c & 1-c \end{pmatrix} \begin{pmatrix} x_n \\ s_n \end{pmatrix}.$$

Calculate the determinant. What does the determinant say about area in the (x, s) plane? Then eliminate s_n and obtain a second-order recurrence for x_n .

- (f) Implement the update in MATLAB for $N = 40$ steps. Make two plots for each case: x_n versus n , and s_n versus x_n .

- (i) $K = 0$, with $x_0 = 0$, $s_0 = 1$;
- (ii) $K = -1$, with $x_0 = 0$, $s_0 = 6$;
- (iii) $K = -cx_{n+1}$, with $x_0 = 1$, $s_0 = 0$, $c = 0.25$.

Briefly describe what structure you see in each case.

- (g) Now explore a second update convention. In the rule used above, s_n is the *outgoing* step from x_n to x_{n+1} : first move with s_n , then compute the turn at the new point. For this part only, shift the step index by one and reinterpret s_n as the *incoming* step that brought the system to x_n ,

$$s_n = x_n - x_{n-1}.$$

At x_n , first apply the turn and then take the new step:

$$\begin{aligned} s_{n+1} &= s_n + K(x_n, s_n, t_n), \\ x_{n+1} &= x_n + s_{n+1}. \end{aligned}$$

Draw a vector picture of one update for each convention. In each picture label x_n , x_{n+1} , the old step, the turn K , and the new step. Explain in words where the turn is applied in the two rules.

- (h) For the alternative rule, derive s_n and x_n for:

- (i) $K = 0$;
- (ii) constant $K = K_0$.

Compare the constant-turn result with the original rule. Specialize to $K_0 = -1$.

- (i) Apply the alternative rule to the spring, now using

$$K(x_n) = -cx_n.$$

Show that

$$\begin{pmatrix} x_{n+1} \\ s_{n+1} \end{pmatrix} = \begin{pmatrix} 1-c & 1 \\ -c & 1 \end{pmatrix} \begin{pmatrix} x_n \\ s_n \end{pmatrix}.$$

Compute the determinant and eliminate s_n to obtain a second-order recurrence for x_n . Compare the recurrence with the one from the original update rule.

Then modify your MATLAB code so that it implements *both* update rules for the same initial values. For $c = 0.25$, $x_0 = 1$, and $s_0 = 0$, plot both trajectories in the (x, s) plane on the same axes. What changes, and what does not?

Optional analytic check. Verify directly that the original spring map preserves

$$I_A = s^2 + cxs + cx^2,$$

while the alternative map preserves

$$I_B = s^2 - cxs + cx^2.$$

Use these expressions to explain the opposite tilts of the phase-space curves.

- (j) Finally, use the two constant-acceleration updates as a scale estimate. Let

$$K = a\Delta^2, \quad s_0 = v_0\Delta, \quad T = n\Delta.$$

Show that at the same elapsed time T the two schemes predict

$$x_A(T) = x_0 + v_0T + \frac{1}{2}a(T^2 - T\Delta),$$

and

$$x_B(T) = x_0 + v_0T + \frac{1}{2}a(T^2 + T\Delta).$$

Hence find their separation $|x_B - x_A|$. How does each compare with the exact continuum constant-acceleration result?

As a thought experiment, take Δ to be the Planck time,

$$t_P \approx 5.39 \times 10^{-44} \text{ s},$$

and use $a = g = 9.81 \text{ m/s}^2$ for $T = 1 \text{ s}$. Estimate the difference between the two discrete predictions. For scale, the Planck length is approximately $\ell_P = 1.62 \times 10^{-35} \text{ m}$. What fraction of a Planck length is the difference?

Interpret this only as a discretization scale estimate, not as evidence that physical time is discrete or that either update rule is fundamental.

Problem 6

Challenge

Before varying an entire trajectory, it is useful to remember what “stationary” means for an ordinary function. A stationary point need not be a minimum.

- (a) Consider

$$V(x) = x^4 - 2x^2.$$

Find all stationary points and classify them. Why would it be misleading to say that the condition $dV/dx = 0$ always finds a minimum?

- (b) Now consider the two-variable function

$$V(x, y) = x^2 - y^2.$$

Let $\mathbf{q} = (x, y)^T$ and $\delta\mathbf{q} = (\delta x, \delta y)^T$. Show that

$$\delta V = (\nabla V)^T \delta\mathbf{q}.$$

Show that the origin is stationary under every infinitesimal variation but is neither a maximum nor a minimum. Explain how the two coordinate directions reveal the saddle.

- (c) The length of a plane curve written as $x = x(y)$ between fixed endpoint values (x_1, y_1) and (x_2, y_2) is

$$\ell[x] = \int_{y_1}^{y_2} \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy.$$

What is being varied in this problem: a number, a point, or a whole function? If $x(y) \mapsto x(y) + \epsilon \eta(y)$, what endpoint conditions should $\eta(y)$ satisfy when the endpoints are fixed? Why is the notation $\delta \ell = 0$ the natural analog of $dV/dx = 0$?

- (d) Suppose the local speed of light in a medium is $u(x, y)$. The travel time for the same curve can be written

$$T[x] = \int_{y_1}^{y_2} \frac{\sqrt{1 + (dx/dy)^2}}{u(x, y)} dy.$$

Identify the function $f(x, x', y)$ in the generic functional

$$\mathcal{J}[x] = \int f(x, x', y) dy, \quad x' = \frac{dx}{dy}.$$

What happens when u is constant? Why is “stationary travel time” more precise than saying that light always follows a minimum of $T[x]$?

- (e) Compare the three levels

$$\frac{dV}{dx} = 0, \quad \delta V = (\nabla V)^T \delta \mathbf{q} = 0, \quad \delta \mathcal{J}[x] = 0.$$

What is being varied in each case, and why does the last condition suggest that a global-looking integral can generate a local differential equation along a path?

If vector or matrix differentiation feels rusty, this question may be useful. You do not need to turn in background questions.

Problem 7

Background

Throughout this problem, vectors are column vectors. For a differentiable scalar function $f(\mathbf{x})$, define the gradient by

$$df = (\nabla_{\mathbf{x}} f)^T d\mathbf{x}.$$

Thus the gradient is the vector that represents the first-order change in f under an arbitrary displacement $d\mathbf{x}$.

This problem is a background check on vector and matrix differentiation. The results will be used repeatedly in mechanics, especially when forces are obtained from potentials.

- (a) Let $\mathbf{a} \in \mathbb{R}^n$ be constant and

$$f(\mathbf{x}) = \mathbf{a}^T \mathbf{x}.$$

Find $\nabla_{\mathbf{x}} f$.

- (b) Let

$$f(\mathbf{x}) = \mathbf{x}^T \mathbf{x} = \|\mathbf{x}\|^2.$$

Show that

$$\nabla_{\mathbf{x}} f = 2\mathbf{x}.$$

- (c) Let B be a constant $n \times n$ matrix and

$$f(\mathbf{x}) = \frac{1}{2} \mathbf{x}^T B \mathbf{x}.$$

Do not assume that B is symmetric.

- (i) Show that

$$\nabla_{\mathbf{x}} f = \frac{1}{2} (B + B^T) \mathbf{x}.$$

- (ii) What does the result become when $B = B^T$? What is the Hessian $H_{ij} = \partial^2 f / (\partial x_i \partial x_j)$ in that case?

- (d) Let A be an $m \times n$ matrix, $\mathbf{b} \in \mathbb{R}^m$, and let $C = C^T$ be an $m \times m$ matrix. Define

$$f(\mathbf{x}) = \frac{1}{2} (A\mathbf{x} - \mathbf{b})^T C (A\mathbf{x} - \mathbf{b}).$$

Show that

$$\nabla_{\mathbf{x}} f = A^T C (A\mathbf{x} - \mathbf{b}).$$

- (e) Apply the previous result to the discrete spring energy

$$U(\mathbf{u}) = \frac{1}{2} (A\mathbf{u})^T C (A\mathbf{u}),$$

where A is an incidence/difference matrix and $C = C^T$ is the constitutive matrix.

- (i) Find $\nabla_{\mathbf{u}} U$ and the internal restoring force $\mathbf{F}_{\text{int}}$.
(ii) What is the Hessian of U ?
(f) Finally, let

$$\mathbf{l} = \mathbf{x}_2 - \mathbf{x}_1, \quad l = \|\mathbf{l}\|, \quad l \neq 0.$$

Use the differential definition of the gradient to show that

$$\nabla_{\mathbf{x}_1} l = -\hat{\mathbf{l}}, \quad \nabla_{\mathbf{x}_2} l = +\hat{\mathbf{l}},$$

where $\hat{\mathbf{l}} = \mathbf{l}/l$.