

Exam I Review

Lectures 1–8: structure, equations, and connections

PHYS V1100: Analytical Dynamics — Fall 2026

How to use this review

This is a supplement to the lecture and student notes, not a replacement for them. It is intended for a student who has already followed the course and wants to recover the main structure quickly before Exam I. Focus on being able to explain *why* each equation has the form it does, what information is encoded by the notation, what assumptions are being made, and how the discrete and continuum viewpoints connect.

The arc of Lectures 1–8

state and update $\longrightarrow$ local differences and boundaries $\longrightarrow$ Newtonian mechanics $\longrightarrow$ discrete action

Hamilton's principle $\longrightarrow$ generalized coordinates $\longrightarrow$ symmetry and Noether
 $\longrightarrow$ normal modes and discrete fields

A recurring course theme is that a compact equation becomes meaningful only after its structure is visible. The same ideas reappear in several forms: node–edge maps become derivatives, transposes become adjoints and integration by parts, nullspaces become symmetries and zero modes, and changes of basis become normal-mode or spectral representations.

Lecture 1 — State, dynamics, and discrete structure

Main point

A state must contain enough information for the dynamics to close. A trajectory is the history generated from an initial state and an evolution law.

The basic update picture is

$$S_{n+1} = F(S_n).$$

A reduced description $s = P(S)$ need not have its own autonomous update law: two different complete states can project to the same reduced state and then evolve differently. Thus “what is the state?” is a dynamical question, not just a bookkeeping choice.

Discrete geometry begins by separating quantities that live on *nodes* from changes that live on *edges*. For an ordered chain,

$$\Delta q_n = q_{n+1} - q_n.$$

Summing local differences telescopes to a boundary difference,

$$\sum_n \Delta q_n = q_{\text{final}} - q_{\text{initial}}.$$

On a ring the boundary disappears and the total oriented difference around the cycle is zero. Applying a first-difference structure twice leads naturally to a second difference, the discrete precursor of a second derivative.

You should be able to explain

Why x alone is not a complete state for $m\ddot{x} = F(x, \dot{x}, t)$; why $(x, \dot{x})$ is; and why a chain and a ring can have the same local difference rule but different global structure.

Lecture 2 — Incidence, stiffness, boundaries, and $A^T C A$

For a spring network, the incidence matrix A maps nodal displacements to oriented edge extensions,

$$\mathbf{e} = A\mathbf{u}.$$

The constitutive matrix C converts extensions to edge forces. The elastic energy is

$$U = \frac{1}{2} \mathbf{e}^T C \mathbf{e} = \frac{1}{2} \mathbf{u}^T A^T C A \mathbf{u},$$

so

$$K = A^T C A, \quad \mathbf{F}_{\text{int}} = -A^T C A \mathbf{u}.$$

The structure is worth remembering as a sequence,

$$\text{nodes } \mathbf{u} \xrightarrow{A} \text{edge changes } \mathbf{e} \xrightarrow{C} \text{edge forces} \xrightarrow{A^T} \text{nodal balance}.$$

A contains connectivity and orientation; C contains constitutive information. Boundary conditions alter the global operator even when the interior nearest-neighbor rule is unchanged. On a ring,

$$A\mathbf{1} = 0,$$

so uniform translation lies in the nullspace. Fixed boundaries remove that rigid-translation zero mode.

The identity

$$\mathbf{v}^T A \mathbf{u} = \mathbf{u}^T A^T \mathbf{v}$$

becomes discrete summation by parts. Interior components of $A^T \mathbf{v}$ act like a negative derivative or divergence, while endpoint components carry the boundary terms. This is the finite-dimensional structure that later becomes integration by parts.

Lecture 3 — Newtonian mechanics, work, energy, and linearization

For a particle,

$$m\ddot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}, t).$$

A second-order equation can be rewritten as a first-order system for the complete state $(\mathbf{x}, \dot{\mathbf{x}})$. The exact flow over a time interval is an update map; an explicit numerical step is generally only an approximation to that flow.

For many particles, internal forces cancel from the center-of-mass equation when Newton's third law applies,

$$M\ddot{\mathbf{X}}_{\text{cm}} = \sum_i \mathbf{F}_i^{\text{ext}}.$$

Power and work connect force to energy,

$$\frac{dT}{dt} = \mathbf{F} \cdot \dot{\mathbf{x}}.$$

For a conservative force derived from a time-independent potential,

$$\boxed{\mathbf{F} = -\frac{\partial V}{\partial \mathbf{x}}}, \quad \boxed{\frac{d}{dt}(T + V) = 0}.$$

If the potential has explicit time dependence, the force may still be conservative in the spatial sense while the mechanical energy need not be conserved. Near an equilibrium, the force may be linearized. The Hessian of the potential becomes the stiffness matrix; for the spring-network example,

$$H = A^T C A,$$

which reconnects Newtonian linearization to Lecture 2.

Important distinction

A nonlinear mechanical system can have a linearized approximation near equilibrium. The approximation belongs to the expansion around the equilibrium, not to Newton's law itself.

Lecture 4 — From discrete updates to a discrete action

Lecture 4 put two descriptions side by side:

$$\boxed{\text{Newton: initial-value march}} \quad \longleftrightarrow \quad \boxed{\text{Hamilton: compare whole paths with fixed boundaries.}}$$

For a discrete path $x_0, \dots, x_N$ with step Δt , a discrete action has the form

$$\boxed{S_d = \sum_{n=0}^{N-1} L_d(x_n, x_{n+1}).}$$

An interior value x_n appears only in its two adjacent segments. Varying that one node gives a local discrete Euler–Lagrange equation. For

$$L_d(x_n, x_{n+1}) = \frac{m}{2\Delta t} (x_{n+1} - x_n)^2 - \Delta t V(x_n),$$

one obtains

$$\boxed{m \frac{x_{n+1} - 2x_n + x_{n-1}}{\Delta t^2} = - \left. \frac{\partial V}{\partial x} \right|_{x=x_n}}.$$

Two statements must be kept separate:

- the difference equation is *exact* for the chosen discrete action;
- identifying the second difference with $\ddot{x}$ is a continuum approximation for a sufficiently smooth trajectory as Δt becomes small.

The discrete equation suggests generalized momentum,

$$p = \frac{\partial L}{\partial \dot{x}},$$

and the mechanical choice

$$\boxed{L = T - V}$$

recovers Newton's equation.

Lecture 5 — Hamilton's principle and Euler–Lagrange

A neighboring path is written

$$x_\epsilon(t) = x(t) + \epsilon\eta(t), \quad \eta(t_1) = \eta(t_2) = 0.$$

Hamilton's principle is

$$\boxed{\delta S = 0}, \quad S[x] = \int_{t_1}^{t_2} L(x, \dot{x}, t) dt.$$

The first variation is

$$\delta S = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial x} \delta x + \frac{\partial L}{\partial \dot{x}} \delta \dot{x} \right] dt.$$

Integration by parts moves the time derivative off the variation,

$$\delta S = \left[\frac{\partial L}{\partial \dot{x}} \delta x \right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial x} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) \right] \delta x dt.$$

Fixed endpoint variations remove the boundary term. Because the interior variation is arbitrary and can be localized, the coefficient must vanish pointwise:

$$\boxed{\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0.}$$

For many coordinates,

$$\boxed{\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = 0.}$$

The key conceptual chain is

$$\boxed{\text{variation} \rightarrow \text{move the derivative} \rightarrow \text{boundary} + \text{interior} \rightarrow \text{local equation}.}$$

Lecture 6 — Constraints and generalized coordinates

For N particles in three dimensions with m independent holonomic constraints,

$$f_\alpha(\mathbf{r}_1, \dots, \mathbf{r}_N, t) = 0, \quad s = 3N - m$$

independent generalized coordinates remain locally. A coordinate map is

$$\boxed{\mathbf{r}_i = \mathbf{r}_i(\mathbf{q}, t).}$$

Differentiating gives

$$\boxed{\dot{\mathbf{r}}_i = \sum_j \frac{\partial \mathbf{r}_i}{\partial q_j} \dot{q}_j + \frac{\partial \mathbf{r}_i}{\partial t}.$$

The geometry of the coordinate map therefore determines the kinetic energy.

An actual differential follows the motion in time,

$$d\mathbf{r}_i = \sum_j \frac{\partial \mathbf{r}_i}{\partial q_j} dq_j + \frac{\partial \mathbf{r}_i}{\partial t} dt,$$

whereas a virtual displacement compares nearby allowed configurations at the same time,

$$\boxed{\delta t = 0}, \quad \boxed{\delta \mathbf{r}_i = \sum_j \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j.}$$

For ideal constraints, the reaction forces do no virtual work. Projecting Newton's law onto allowed virtual displacements leads to generalized forces and then to Lagrange's equations. For a conservative system,

$$\boxed{L(\mathbf{q}, \dot{\mathbf{q}}, t) = T - V}$$

contains the constrained dynamics without solving explicitly for the ideal reaction forces.

A cyclic coordinate does not appear explicitly in L . Its conjugate momentum is therefore constant. This is the first clue that conservation laws are tied to symmetry.

Lecture 7 — Constraint multipliers, symmetry, and Noether

For a holonomic constraint $F(\mathbf{q}, t) = 0$, the reaction force lies normal to the constraint surface. Once that geometric fact is understood, the constraint force can be packaged into an augmented Lagrangian,

$$\boxed{L_\lambda = L + \lambda F.}$$

The Euler–Lagrange equations for L_λ , together with variation with respect to λ , give the constrained equations and recover $F(\mathbf{q}, t) = 0$. The multiplier is therefore a compact way to represent a reaction whose direction came first from the constraint geometry and virtual work.

For a generalized coordinate,

$$p_j = \frac{\partial L}{\partial \dot{q}_j}, \quad \dot{p}_j = \frac{\partial L}{\partial q_j}.$$

If q_k is cyclic, then p_k is conserved. The deeper statement is that a cyclic coordinate is one representation of a continuous symmetry.

For an infinitesimal fixed-time symmetry

$$\delta_s q_n = \epsilon h_n(\mathbf{q}, t),$$

the on-shell variation of the action reduces to the endpoint term,

$$\delta_s S = \left[\sum_n p_n \delta_s q_n \right]_{t_1}^{t_2}.$$

If the action is invariant, $\delta_s S = 0$, then

$$\boxed{J = \sum_n p_n h_n = \text{constant}.}$$

Thus

$$\boxed{\text{equations of motion} + \text{continuous symmetry} \implies \text{conserved quantity}.}$$

Important examples:

- translation symmetry $\implies$ total linear momentum;

- rotation symmetry $\Rightarrow$ angular momentum;
- no explicit time dependence $\Rightarrow$ conserved Lagrangian energy

$$E_L = \sum_j p_j \dot{q}_j - L, \quad \frac{dE_L}{dt} = -\frac{\partial L}{\partial t};$$

for ordinary mechanical $L = T - V$, $E_L = T + V$;

- if $\delta_s L = \epsilon dB/dt$, then the quasi-symmetry charge is

$$\sum_j p_j h_j - B = \text{constant.}$$

The boundary term that was set to zero in the fixed-endpoint derivation of Euler–Lagrange is precisely what carries the Noether charge when a symmetry moves the endpoint configurations.

Lecture 8 — Coupled oscillators, modes, and discrete fields

A list of values $q_1, \dots, q_N$ does not by itself say whether the variables live on a chain, ring, surface, or other structure. The interaction/incidence data supply the meaning of “local.” One can view a discrete field either as a sampling/interpolation of an underlying continuum or as a genuinely discrete interacting system whose local graph/simplicial structure determines an effective geometry. In higher dimensions, physical interaction edges and geometric simplicial completion need not be identical.

For the equal-mass spring chain,

$$\mathbf{e} = A\mathbf{q}, \quad V = \frac{\kappa}{2} \mathbf{e}^T \mathbf{e} = \frac{\kappa}{2} \mathbf{q}^T A^T A \mathbf{q}.$$

Define

$$K = A^T A, \quad m\ddot{\mathbf{q}} = -\kappa K \mathbf{q}.$$

Here K is the dimensionless equal-spring operator; the corresponding physical stiffness matrix is $\kappa K = \kappa A^T A$. A normal mode has the form

$$\mathbf{q}(t) = \mathbf{u}_\alpha a_\alpha(t), \quad K \mathbf{u}_\alpha = \lambda_\alpha \mathbf{u}_\alpha,$$

so

$$\omega_\alpha^2 = \frac{\kappa}{m} \lambda_\alpha.$$

For a fixed–fixed chain with N moving sites,

$$Q_{nk} = \sqrt{\frac{2}{N+1}} \sin\left(\frac{nk\pi}{N+1}\right), \quad n, k = 1, \dots, N,$$

and

$$\lambda_k = 2 \left[1 - \cos\left(\frac{k\pi}{N+1}\right) \right].$$

The orthogonal change of coordinates

$$\mathbf{q} = Q\mathbf{a}, \quad \mathbf{a} = Q^T \mathbf{q}$$

is a discrete sine transform (DST-I, up to indexing/scaling convention). In mode coordinates the quadratic Lagrangian separates into independent oscillators.

For a periodic ring, uniform displacement is a symmetry,

$$\mathbf{q} \rightarrow \mathbf{q} + c\mathbf{1},$$

so

$$K\mathbf{1} = 0, \quad \lambda_0 = 0, \quad \omega_0 = 0.$$

The periodic eigenvectors are Fourier modes $u_n = e^{ikna}$ with

$$\lambda(k) = 2 - 2\cos(ka) = 4\sin^2(ka/2).$$

The ring is naturally diagonalized by a discrete Fourier transform. Fixed and periodic systems can share the same local second-difference rule while their boundary conditions select different allowed wave numbers and global mode bases.

Do not confuse these

A system with N coordinates has N normal modes counting multiplicity. Degenerate modes may share one frequency. A zero mode is still a mode; it corresponds to a symmetry direction rather than an oscillation with nonzero frequency.

Connections worth being able to say out loud

Pair of ideas	What connects them
State vs. trajectory	The state is the information needed to continue the evolution; the trajectory is the history generated by repeated or continuous evolution.
Nodes vs. edges	State-like values can live on nodes; local differences/interactions live on edges.
A vs. A^T	A forms local differences; A^T returns edge quantities to nodal balance and carries boundary structure.
Discrete vs. continuum	A discrete equation can be exact for a discrete model; interpreting differences as derivatives introduces a continuum approximation.
Force vs. constraint	A spring supplies a force and normally does not reduce configuration dimension; a rigid geometric condition is a constraint.
$d\mathbf{r}$ vs. $\delta\mathbf{r}$	$d\mathbf{r}$ follows an actual time evolution; $\delta\mathbf{r}$ compares allowed configurations at fixed time.
Fixed-endpoint variation vs. symmetry variation	The former kills the action boundary term; the latter generally moves the endpoints, and that boundary term becomes the Noether charge.
Symmetry vs. conservation	Continuous action symmetry plus the equations of motion gives a conserved Noether quantity.
Eigenvectors vs. mode coordinates	Eigenvectors define a new basis; the coefficients in that basis are the independent modal coordinates.
Fixed chain vs. ring	Same local operator can coexist with different topology/boundaries, allowed wave numbers, zero modes, and spectral transforms.

Compact equation list

$$\begin{array}{lll}
S_{n+1} = F(S_n), & \mathbf{e} = A\mathbf{q}, & K = A^T C A, \\
m\ddot{\mathbf{x}} = \mathbf{F}, & \mathbf{F} = -\frac{\partial V}{\partial \mathbf{x}}, & \frac{d}{dt}(T + V) = 0 \quad (\text{conservative, autonomous}), \\
S = \int L dt, & L = T - V, & \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = 0, \\
p_j = \frac{\partial L}{\partial \dot{q}_j}, & E_L = \sum_j p_j \dot{q}_j - L, & \frac{dE_L}{dt} = -\frac{\partial L}{\partial t}, \\
\delta_s q_j = \epsilon h_j, & J = \sum_j p_j h_j, & \delta_s L = \epsilon \frac{dB}{dt} \Rightarrow J - B = \text{constant}, \\
M\ddot{\mathbf{q}} + K\mathbf{q} = 0, & (K - \omega^2 M)\mathbf{u} = 0, & \omega_\alpha^2 = \frac{\kappa}{m} \lambda_\alpha \quad \text{when } m\ddot{\mathbf{q}} = -\kappa(A^T A)\mathbf{q}.
\end{array}$$

Selected review problems

The following are shortened selections from existing QABank problems. They are included because each one tests a recurring structural idea rather than a special trick. Try them before turning to the solutions.

Problem 1 — Incidence, stiffness, and boundary structure

Based on Q02.01 and Q02.04.

Four free masses lie between two fixed walls. With equal spring stiffness set to one, use

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

- What does $A\mathbf{u}$ represent physically? What information is carried by A rather than by $\mathbf{u}$?
- Write the elastic energy and identify the stiffness matrix.
- Why is there no rigid-translation zero mode for this fixed-fixed system?

Problem 2 — Discrete action and the continuum interpretation

Based on Q04.03.

For

$$L_d(x_n, x_{n+1}) = \frac{m}{2\Delta t} (x_{n+1} - x_n)^2 - \Delta t V(x_n),$$

the discrete Euler–Lagrange equation is

$$D_2 L_d(x_{n-1}, x_n) + D_1 L_d(x_n, x_{n+1}) = 0.$$

(a) Show that

$$m \frac{x_{n+1} - 2x_n + x_{n-1}}{\Delta t^2} = - \left. \frac{\partial V}{\partial x} \right|_{x=x_n}.$$

(b) Which statement is exact: the difference equation above, or the identification of the second difference with $\ddot{x}(t_n)$? Explain.

Problem 3 — Rotation symmetry and Noether

A particle of mass m moves in the xy plane in a central potential,

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - U(r), \quad r = \sqrt{x^2 + y^2}.$$

Consider the infinitesimal rotation

$$\delta_s x = -\epsilon y, \quad \delta_s y = \epsilon x.$$

(a) Show directly that the kinetic energy and r are unchanged to first order in ϵ , and hence that $\delta_s L = 0$.

(b) Use Noether's theorem to identify the conserved charge and interpret it physically.

(c) Now add

$$V_{\text{ext}} = \frac{1}{2}\kappa_x x^2 + \frac{1}{2}\kappa_y y^2, \quad \kappa_x \neq \kappa_y.$$

Does continuous rotation symmetry survive? What happens to the conservation law from part (b)?

Problem 4 — Fixed-chain normal modes

Based on Q08.04.

For the N -site fixed-fixed chain,

$$(K\mathbf{y})_j = 2y_j - y_{j-1} - y_{j+1}, \quad y_0 = y_{N+1} = 0.$$

Take

$$y_j^{(r)} = \sin(j\theta_r), \quad \theta_r = \frac{r\pi}{N+1}.$$

(a) Show that $\mathbf{y}^{(r)}$ satisfies the fixed endpoint conditions.

(b) Verify that it is an eigenvector and find

$$\lambda_r = 2(1 - \cos \theta_r).$$

(c) For identical masses m and spring stiffness κ , find ω_r .

Problem 5 — Generalized coordinate from a constraint

A bead of mass m moves without friction on a fixed straight wire making an angle α above the horizontal. Let s measure signed distance along the wire from a chosen origin, so

$$\mathbf{r}(s) = \begin{pmatrix} s \cos \alpha \\ s \sin \alpha \end{pmatrix}.$$

Gravity acts downward, and a spring along the wire contributes the potential $\frac{1}{2}\kappa s^2$.

- (a) Use the coordinate map to find T , V , and $L(s, \dot{s})$.
- (b) Derive the equation of motion for $s(t)$ from the Euler–Lagrange equation.
- (c) Is s cyclic? Is the Lagrangian autonomous? Write the corresponding conserved energy.
- (d) Why does the normal reaction force from the wire not appear in this reduced equation?

Solutions to the selected review problems

Problem 1 solution

$A\mathbf{u}$ is the vector of oriented spring extensions. The entries of $\mathbf{u}$ are nodal displacement values; A specifies which nodes are connected, their orientation, and how the fixed boundaries enter the extension map.

With unit spring stiffness,

$$U = \frac{1}{2}(A\mathbf{u})^T(A\mathbf{u}) = \frac{1}{2}\mathbf{u}^T A^T A \mathbf{u}, \quad K = A^T A.$$

There is no rigid-translation zero mode because the end nodes are fixed. A uniform shift of all four free masses stretches the two boundary springs, so $A\mathbf{1} \neq 0$.

Problem 2 solution

For the incoming segment,

$$D_2 L_d(x_{n-1}, x_n) = \frac{m}{\Delta t}(x_n - x_{n-1}).$$

For the outgoing segment,

$$D_1 L_d(x_n, x_{n+1}) = -\frac{m}{\Delta t}(x_{n+1} - x_n) - \Delta t \left. \frac{\partial V}{\partial x} \right|_{x=x_n}.$$

Setting their sum equal to zero and dividing by Δt gives

$$\boxed{m \frac{x_{n+1} - 2x_n + x_{n-1}}{\Delta t^2} = - \left. \frac{\partial V}{\partial x} \right|_{x=x_n}}.$$

This equation is exact for the stated discrete action. The approximation enters only when the second difference is interpreted as an approximation to $\ddot{x}(t_n)$ for a smooth continuum path.

Problem 3 solution

The velocity variations are

$$\delta_s \dot{x} = -\epsilon \dot{y}, \quad \delta_s \dot{y} = \epsilon \dot{x},$$

so

$$\delta_s T = m \dot{x} \delta_s \dot{x} + m \dot{y} \delta_s \dot{y} = 0.$$

Also,

$$\delta_s (x^2 + y^2) = 2x \delta_s x + 2y \delta_s y = 0,$$

so r and therefore $U(r)$ are unchanged. Hence $\delta_s L = 0$.

The generators are $h_x = -y$ and $h_y = x$. Noether's theorem gives

$$J = p_x h_x + p_y h_y = -y p_x + x p_y = \boxed{x p_y - y p_x},$$

which is the z component of angular momentum.

For the anisotropic quadratic potential,

$$\delta_s V_{\text{ext}} = \epsilon (\kappa_y - \kappa_x) xy,$$

which is not generally zero when $\kappa_x \neq \kappa_y$. Continuous rotation symmetry is therefore broken, and $x p_y - y p_x$ is not generally conserved.

Problem 4 solution

The endpoint values are

$$y_0^{(r)} = 0, \quad y_{N+1}^{(r)} = \sin(r\pi) = 0.$$

For an interior site,

$$\begin{aligned} (K \mathbf{y}^{(r)})_j &= 2 \sin(j\theta_r) - \sin((j-1)\theta_r) - \sin((j+1)\theta_r) \\ &= 2(1 - \cos \theta_r) \sin(j\theta_r). \end{aligned}$$

Thus

$$\boxed{\lambda_r = 2(1 - \cos \theta_r) = 4 \sin^2(\theta_r/2)}.$$

With

$$m \ddot{\mathbf{q}} = -\kappa K \mathbf{q},$$

a mode amplitude obeys

$$\ddot{a}_r + \frac{\kappa}{m} \lambda_r a_r = 0,$$

so

$$\boxed{\omega_r = 2 \sqrt{\frac{\kappa}{m}} \sin \left(\frac{r\pi}{2(N+1)} \right)}.$$

Problem 5 solution

Differentiating the coordinate map gives

$$\dot{\mathbf{r}} = \begin{pmatrix} \dot{s} \cos \alpha \\ \dot{s} \sin \alpha \end{pmatrix}, \quad \dot{\mathbf{r}}^2 = \dot{s}^2.$$

Therefore

$$T = \frac{1}{2}m\dot{s}^2.$$

Since $y = s \sin \alpha$,

$$V = mgs \sin \alpha + \frac{1}{2}\kappa s^2,$$

and

$$L = \frac{1}{2}m\dot{s}^2 - mgs \sin \alpha - \frac{1}{2}\kappa s^2.$$

The Euler–Lagrange equation gives

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{s}} \right) - \frac{\partial L}{\partial s} = 0,$$

so

$$m\ddot{s} + \kappa s + mg \sin \alpha = 0.$$

The coordinate s is not cyclic, but L has no explicit time dependence. Thus

$$E = \frac{1}{2}m\dot{s}^2 + mgs \sin \alpha + \frac{1}{2}\kappa s^2 = \text{constant}.$$

The wire's ideal normal reaction does no virtual work along the allowed displacement. Using s as the generalized coordinate therefore absorbs the geometric constraint and removes that reaction force from the reduced equation.

Final review question

For each major equation in this document, ask yourself three things: **What are the objects? What structure or assumption produced the equation? What would change if the boundary condition, coordinates, or symmetry changed?** If you can answer those questions, the equation is no longer just something to remember.