PHYS V1100: Analytical Dynamics
Lecture 16: Hamiltonian Mechanics II

Poisson brackets, generators, conservation, and phase-space flow

1 From the motion of a state to the change of an observable

Lecture 15 replaced velocities by conjugate momenta and obtained Hamilton's equations,

$$\dot{\boldsymbol{q}} = \frac{\partial H}{\partial \boldsymbol{p}}, \qquad \dot{\boldsymbol{p}} = -\frac{\partial H}{\partial \boldsymbol{q}}. \tag{1}$$

For a regular system with N independent coordinates, $(\boldsymbol{q}, \boldsymbol{p})$ gives a point in a $2N$-dimensional phase space. The Hamiltonian assigns a velocity to that point. Given initial data and the usual local existence and uniqueness conditions, the state follows a trajectory.

But the coordinates themselves are not the only quantities whose changes interest us. An angular momentum, an energy, or a product such as $q_i p_j$ is also a function on phase space. Let

$$F = F(\boldsymbol{q}, \boldsymbol{p}, t)$$

be any differentiable scalar function. Calling it an *observable* here means that it assigns a scalar value to a state; it need not be conserved or even have a special physical name.

There are two distinct reasons its value can change. The state can move to a point where F has a different value, and the function assigning values can itself depend explicitly on time. The ordinary chain rule keeps these contributions separate:

$$\begin{aligned} \dot{F} &= \left(\frac{\partial F}{\partial \boldsymbol{q}}\right)^T \dot{\boldsymbol{q}} + \left(\frac{\partial F}{\partial \boldsymbol{p}}\right)^T \dot{\boldsymbol{p}} + \frac{\partial F}{\partial t} \\ &= \left(\frac{\partial F}{\partial \boldsymbol{q}}\right)^T \frac{\partial H}{\partial \boldsymbol{p}} - \left(\frac{\partial F}{\partial \boldsymbol{p}}\right)^T \frac{\partial H}{\partial \boldsymbol{q}} + \frac{\partial F}{\partial t}. \end{aligned} \tag{2}$$

All vector derivatives of scalars are columns, so the transposes produce scalar pairings. The overdot is the total derivative along a trajectory; $\partial F/\partial t$ is taken at fixed $(\boldsymbol{q}, \boldsymbol{p})$. They are not interchangeable.

The first two terms in (2) have a useful structure. They involve two functions in an antisymmetric way: exchanging F and H reverses the sign. We will first name this operation for an arbitrary pair of functions. Only then will we choose one of them to generate a particular motion.

2 The Poisson bracket: one operation on two functions

For scalar phase-space functions F and G, define

$$\boxed{\{F, G\} = \sum_{i=1}^{N} \left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} \right).} \tag{3}$$

Equivalently,

$$\{F, G\} = \left(\frac{\partial F}{\partial \boldsymbol{q}}\right)^T \frac{\partial G}{\partial \boldsymbol{p}} - \left(\frac{\partial F}{\partial \boldsymbol{p}}\right)^T \frac{\partial G}{\partial \boldsymbol{q}}.$$

Time, when present, is a parameter in this operation. The bracket contains derivatives with respect to the canonical phase-space coordinates, not time derivatives. Its result is generally another function on phase space, not a constant.

The bracket is therefore a *relation between two functions*. Saying that a function "has zero bracket" is incomplete: with which other function? In particular, choosing $G = H$ turns (2) into

$$\boxed{\dot{F} = \{F, H\} + \frac{\partial F}{\partial t}.} \tag{4}$$

The term $\{F, H\}$ measures the change of F caused by Hamiltonian motion through phase space. It is not the full time derivative when F has explicit time dependence.

2.1 Coordinate functions and the sign convention

The coordinate functions have particularly simple derivatives:

$$\frac{\partial q_i}{\partial q_k} = \delta_{ik}, \qquad \frac{\partial q_i}{\partial p_k} = 0, \qquad \frac{\partial p_j}{\partial p_k} = \delta_{jk}, \qquad \frac{\partial p_j}{\partial q_k} = 0.$$

Substitution in (3) gives

$$\boxed{\{q_i, q_j\} = 0, \qquad \{p_i, p_j\} = 0, \qquad \{q_i, p_j\} = \delta_{ij}.} \tag{5}$$

The order matters: $\{p_i, q_j\} = -\delta_{ij}$. More generally,

$$\{q_i, G\} = \frac{\partial G}{\partial p_i}, \qquad \{p_i, G\} = -\frac{\partial G}{\partial q_i}. \tag{6}$$

Setting $G = H$ recovers both Hamilton equations. Thus the observable evolution law is not an additional dynamical assumption. It contains Hamilton's equations as its coordinate cases and extends them by the chain rule to every other F.

2.2 A calculation to make the operation concrete

For one oscillator,

$$H(q, p) = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2, \qquad m > 0, \quad \omega > 0,$$

take $F(q, p) = qp$. Before inserting a trajectory, differentiate the two functions with their other arguments held fixed:

$$\frac{\partial F}{\partial q} = p, \quad \frac{\partial F}{\partial p} = q, \qquad \frac{\partial H}{\partial p} = \frac{p}{m}, \quad \frac{\partial H}{\partial q} = m\omega^2 q.$$

Then

$$\boxed{\{qp, H\} = \frac{p^2}{m} - m\omega^2 q^2.} \tag{7}$$

Since qp has no explicit time dependence, this is its total time derivative along oscillator motion. We can check without using bracket notation:

$$\frac{d}{dt}(qp) = \dot{q}\, p + q\dot{p} = \frac{p^2}{m} - m\omega^2 q^2.$$

The bracket has packaged the same calculation; it has not bypassed the dynamics. The result is positive where the kinetic contribution dominates and negative where the potential contribution dominates. At an isolated point the two may cancel, but that does not make qp a conserved function throughout phase space.

This also gives a dimensional check. The quantity qp has dimensions of action, and its rate of change has dimensions of energy. Both terms on the right of (7) have those dimensions. There is no reason for the bracket of two arbitrary functions to have the units of either function separately.

3 Conservation is relative to the chosen dynamics

Equation (4) gives the full condition:

$$\boxed{\frac{\partial F}{\partial t} + \{F, H\} = 0.} \tag{8}$$

If this is an identity on a region of phase space and time, F is constant along every solution that stays in that region. A derivative vanishing at one point, or even on one special orbit, is a weaker statement. When F is time independent, the condition reduces to $\{F, H\} = 0$.

3.1 The same momentum, two Hamiltonians

Consider the Cartesian momentum p_x. For

$$H_1 = \frac{p_x^2 + p_y^2}{2m} + V(y),$$

we have $\{p_x, H_1\} = -\partial H_1/\partial x = 0$. The Hamiltonian is independent of x, so the system conserves p_x. For

$$H_2 = \frac{p_x^2 + p_y^2}{2m} + V(x, y),$$

the same function has

$$\{p_x, H_2\} = -\frac{\partial V}{\partial x},$$

which generally does not vanish. Nothing about the definition of p_x changed; the force law changed. A momentum being conjugate to a coordinate is a structural statement, whereas its conservation depends on the Hamiltonian.

For any cyclic coordinate the same argument gives

$$\frac{\partial H}{\partial q_j} = 0 \quad \Longrightarrow \quad \dot{p}_j = 0.$$

It does not imply $\dot{q}_j = 0$. A free particle has conserved momentum and usually changing position.

3.2 An explicitly time-dependent conserved quantity

For the free particle $H = p^2/(2m)$, take

$$F(q, p, t) = q - \frac{t}{m}p.$$

At fixed phase-space coordinates,

$$\frac{\partial F}{\partial t} = -\frac{p}{m}, \qquad \{F, H\} = \frac{p}{m}.$$

These terms are separately nonzero, but they cancel in (8). Along the solution $p = p_0$, $q = q_0 + p_0 t/m$, where the reference time is zero,

$$F(q(t), p(t), t) = q_0.$$

The explicit clock term subtracts the displacement accumulated during free motion. This is a conserved quantity whose bracket with H does *not* vanish.

Conversely, the function $F = t$ has zero bracket with every function because its phase-space derivatives vanish, yet $\dot{F} = 1$. Thus zero bracket and conservation are not equivalent when explicit time dependence is ignored: zero bracket is neither necessary nor sufficient by itself.

3.3 The Hamiltonian itself

Antisymmetry gives $\{H, H\} = 0$, even if H depends explicitly on time. Thus

$$\boxed{\dot{H} = \frac{\partial H}{\partial t}.} \tag{9}$$

An autonomous Hamiltonian is conserved. This result does not require assuming that H has already been identified with a particular mechanical energy. Lecture 15 distinguished that physical identification from the definition of the Hamiltonian.

For a spring with externally prescribed stiffness,

$$H(q, p, t) = \frac{p^2}{2m} + \frac{1}{2}k(t)q^2, \qquad \dot{H} = \frac{1}{2}\dot{k}(t)q^2. \tag{10}$$

Here H is the instantaneous kinetic plus potential energy, but it can change because the external control changes the stiffness. We will return to this example when asking whether energy change implies a change of phase-space volume.

4 The bracket is a derivation, not multiplication

The derivative definition makes several algebraic rules unavoidable. For constants a, b, the bracket is bilinear,

$$\{aF + bG, K\} = a\{F, K\} + b\{G, K\},$$

and antisymmetric,

$$\{F, G\} = -\{G, F\}, \qquad \{F, F\} = 0.$$

The coefficients in this formula are independent of the phase-space coordinates. If they are themselves phase-space functions, their derivatives also contribute. Time-dependent coefficients can be treated as fixed parameters when computing a bracket, but not when taking a total time derivative.

For fixed K, the bracket differentiates a product in exactly the way a directional derivative should:

$$\boxed{\{FG, K\} = F\{G, K\} + G\{F, K\}.} \tag{11}$$

For example, insert $\partial(FG)/\partial q_i = F\,\partial G/\partial q_i + G\,\partial F/\partial q_i$, and the corresponding momentum derivative, into (3). Collecting the terms proportional to F and to G gives (11). Antisymmetry then gives the product rule in the other slot.

For one conjugate pair, the rules yield

$$\{q^2, p^2\} = 2q\{q, p^2\} = 2q(2p\{q, p\}) = 4qp.$$

Direct differentiation gives the same result. Ordinary products commute, $q^2p^2 = p^2q^2$, but their bracket need not vanish. This is not a contradiction: multiplication and the bracket are different operations.

A further identity links nested brackets:

$$\boxed{\{F, \{G, K\}\} + \{G, \{K, F\}\} + \{K, \{F, G\}\} = 0.} \tag{12}$$

This is the *Jacobi identity*. It follows by expanding the derivatives and canceling terms, with the required mixed derivatives commuting. We will use it without a full component proof. Its role is not merely to supply another formula: it constrains how two successive bracket-generated changes fit together. Section 7 uses it to prove a concrete result about conserved quantities.

5 A generator specifies a change of state

The Hamiltonian generates one particular change: physical time evolution. To see the broader structure, let a smooth, time-independent function $G(\boldsymbol{q}, \boldsymbol{p})$ specify an infinitesimal state change at fixed physical time:

$$\delta_G q_i = \epsilon \frac{\partial G}{\partial p_i}, \qquad \delta_G p_i = -\epsilon \frac{\partial G}{\partial q_i}. \tag{13}$$

Here ϵ is the small transformation parameter. The notation δ_G means the particular change generated by G, not an arbitrary variation of an entire path. When the transformation is a symmetry it is a particular symmetry variation, δ_{sym}.

The first-order change in any function follows by Taylor expansion:

$$\begin{aligned} \delta_G F &= \sum_i \left(\frac{\partial F}{\partial q_i} \delta_G q_i + \frac{\partial F}{\partial p_i} \delta_G p_i \right) \\ &= \epsilon \sum_i \left(\frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} \right). \end{aligned}$$

Therefore

$$\boxed{\delta_G F = \epsilon\{F, G\}.} \tag{14}$$

This equation has a direct meaning: the first slot is the quantity being examined, and the second specifies the transformation to which it responds. The formula describes the linear term in ϵ; a finite transformation requires following the generated changes, not simply retaining one first-order step.

For actual time evolution set $G = H$ and $\epsilon = dt$. For a time-independent observable, this gives its whole infinitesimal time change. For $F(\boldsymbol{q}, \boldsymbol{p}, t)$, it gives only the state-motion contribution; the explicit term $(\partial F/\partial t)dt$ must still be included. If H depends on time, it generates the instantaneous state change using its value at that time.

The parameter need not have units of time. For momentum generating a displacement, it has units of length; for angular momentum generating a rotation, it is an angle. In these mechanical examples ϵG has units of action, just as $H dt$ does.

5.1 Translation as a first example

For one Cartesian pair, choose $G = p$. Equation (13) gives

$$\delta_p q = \epsilon, \qquad \delta_p p = 0,$$

so momentum generates a position translation. For an observable,

$$\delta_p F = \epsilon\{F, p\} = \epsilon \frac{\partial F}{\partial q},$$

which is the first-order change in $F(q + \epsilon, p)$. This is an *active* comparison of nearby states in the same coordinates. We have not yet changed the coordinate system used to describe a fixed state; general coordinate changes belong to Lecture 17.

A translation can be generated even when the system is not translation invariant. For an oscillator, translating q changes the potential energy. Being a generator is not the same as generating a symmetry of the particular Hamiltonian.

5.2 Angular momentum generates rotations

Use Cartesian canonical coordinates $(\boldsymbol{x}, \boldsymbol{p})$ and

$$\boldsymbol{L} = \boldsymbol{x} \times \boldsymbol{p}, \qquad L_z = x p_y - y p_x.$$

The momentum derivatives of L_z give the position changes:

$$\{x, L_z\} = \frac{\partial L_z}{\partial p_x} = -y, \qquad \{y, L_z\} = \frac{\partial L_z}{\partial p_y} = x.$$

The position derivatives give the momentum changes, with the essential minus sign:

$$\{p_x, L_z\} = -\frac{\partial L_z}{\partial x} = -p_y, \qquad \{p_y, L_z\} = -\frac{\partial L_z}{\partial y} = p_x.$$

Thus

$$\begin{aligned} \delta_{L_z} x &= -\epsilon y, \quad & \delta_{L_z} y &= \epsilon x, \\ \delta_{L_z} p_x &= -\epsilon p_y, \quad & \delta_{L_z} p_y &= \epsilon p_x. \end{aligned} \tag{15}$$

The z and p_z components do not change. Both position and momentum rotate, as a transformation of the full phase-space state requires.

To verify the direction, take a point on the positive x axis. A positive ϵ moves it toward positive y, a counterclockwise rotation viewed from the positive z axis.

If θ parametrizes the finite generated rotation, the equations

$$\frac{dx}{d\theta} = -y, \qquad \frac{dy}{d\theta} = x$$

have the solution

$$\begin{pmatrix} x(\theta) \\ y(\theta) \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix},$$

with the same matrix acting on (p_x, p_y). The infinitesimal bracket calculation has therefore identified the familiar rotation, not just a formally similar pair of signs.

The corresponding three-dimensional identities are

$$\{x_i, L_j\} = \sum_k \epsilon_{ijk} x_k, \qquad \{p_i, L_j\} = \sum_k \epsilon_{ijk} p_k, \qquad \{L_i, L_j\} = \sum_k \epsilon_{ijk} L_k. \tag{16}$$

Here ϵ_{ijk} is the Levi–Civita symbol, with $\epsilon_{123} = 1$, not the transformation parameter ϵ. One component of the last identity can be checked explicitly. With $L_x = yp_z - zp_y$ and $L_y = zp_x - xp_z$, only the (z, p_z) pair contributes:

$$\{L_x, L_y\} = (-p_y)(-x) - (y)(p_x) = xp_y - yp_x = L_z.$$

The other components follow by cyclic relabeling and antisymmetry. The nonzero brackets express that rotations about different axes have a nontrivial composition law; conserved angular-momentum components need not have zero brackets with one another.

6 When a generated change is a symmetry

For a time-independent generator G, the change of the Hamiltonian under its fixed-time transformation is

$$\delta_G H = \epsilon\{H, G\}.$$

If the Hamiltonian is invariant under that continuous transformation, then $\{H, G\} = 0$. Antisymmetry gives $\{G, H\} = 0$, and (4) then gives $\dot{G} = 0$. In symbols,

$$\boxed{\text{invariance under } G \implies \{H, G\} = 0 \implies \dot{G} = 0, \qquad \frac{\partial G}{\partial t} = 0.} \tag{17}$$

Locally, if the bracket vanishes throughout the region swept out by the generated transformation, H remains constant along that transformation. This is the Hamiltonian version of the symmetry–conservation connection encountered earlier, not a claim that every conceivable state change is a symmetry.

For example, consider a two-dimensional oscillator with

$$H = \frac{p_x^2 + p_y^2}{2m} + \frac{1}{2}k_x x^2 + \frac{1}{2}k_y y^2, \qquad k_x, k_y > 0.$$

Using the derivatives already calculated for L_z,

$$\begin{aligned} \{H, L_z\} &= (k_x x)(-y) + (k_y y)(x) - \left[\frac{p_x}{m}(-p_y) + \frac{p_y}{m}(p_x)\right] \\ &= (k_y - k_x)xy. \end{aligned}$$

The kinetic terms cancel because the ordinary kinetic energy is rotationally invariant. The potential terms cancel throughout phase space only when $k_x = k_y$. Thus the same L_z generates rotations for both oscillators, but is conserved only for the rotationally invariant one:

$$\dot{L}_z = \{L_z, H\} = (k_x - k_y)xy.$$

This also equals the physical torque $xF_y - yF_x$, since $F_x = -k_x x$ and $F_y = -k_y y$. The bracket calculation reproduces the Newtonian interpretation.

The time-independence qualification on G matters. For time-dependent candidate conserved quantities, return to $\partial G/\partial t + \{G, H\} = 0$, as in the free-particle example. The simple fixed-time symmetry argument above is not the full theory of time-dependent canonical transformations, which we do not develop here.

7 Conserved quantities have an algebra

Suppose F and G have no explicit time dependence and are conserved under the same Hamiltonian:

$$\{F, H\} = 0, \qquad \{G, H\} = 0.$$

A sum or product of them is conserved by the ordinary product rule. The less obvious result is that their Poisson bracket is also conserved. Write Jacobi in the equivalent form

$$\{\{F, G\}, H\} + \{\{G, H\}, F\} + \{\{H, F\}, G\} = 0.$$

The last two terms vanish, so

$$\boxed{\{\{F, G\}, H\} = 0.} \tag{18}$$

Since $\{F, G\}$ also has no explicit time dependence, this is precisely its conservation condition. The result is an identity on the relevant phase-space region, not an inference from two accidental zeros on one orbit.

For a rotationally invariant central-force Hamiltonian, all components of $\boldsymbol{L}$ are conserved, and $\{L_x, L_y\} = L_z$ is the concrete example. Taking a bracket need not produce a *new independent* constant of motion; it may produce one already known, a function of the known constants, or zero.

Two functions are said to be *in involution* when

$$\{F, G\} = 0.$$

This is a statement about their mutual bracket, not just their separate conservation under H. The angular-momentum example shows the distinction. By contrast, for two uncoupled oscillators,

$$H = H_1(q_1, p_1) + H_2(q_2, p_2),$$

we have $\{H_1, H_2\} = 0$, because they depend on disjoint conjugate pairs. Both are conserved, and they are in involution. Mutual involution will matter for integrability later; no integrability theorem is being asserted here.

8 From trajectories to a flow of nearby states

Collect the coordinates into a column and name their phase-space velocity:

$$\boldsymbol{z} = \begin{pmatrix} \boldsymbol{q} \\ \boldsymbol{p} \end{pmatrix}, \qquad \dot{\boldsymbol{z}} = \boldsymbol{u}(\boldsymbol{z}, t) = \begin{pmatrix} \partial H/\partial \boldsymbol{p} \\ -\partial H/\partial \boldsymbol{q} \end{pmatrix}. \tag{19}$$

A single solution follows one integral curve of this field. To study a region of phase space, consider many possible initial states and evolve each for the same elapsed time. Their endpoints define a *flow map*

$$\boldsymbol{z}(t) = \Phi_{t,t_0}(\boldsymbol{z}_0).$$

This is the exact finite-time map of the differential equations, not a chosen numerical approximation. For a time-dependent Hamiltonian it generally depends on both t_0 and t, not only their difference.

Assume the Hamiltonian is sufficiently smooth and restrict attention to a common interval on which these solutions exist uniquely. Nearby initial states then have a differentiable flow map. The next question is not whether one state conserves energy, but whether a small *collection of states* expands or contracts under this map.

8.1 What phase-space divergence measures

At a fixed time, write the Jacobian of the velocity field as

$$\mathsf{B}(\boldsymbol{z}, t) = \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{z}^T}, \qquad B_{ab} = \frac{\partial u_a}{\partial z_b}.$$

This is a vector-by-vector derivative, hence the transposed denominator in the course notation. Over a short interval h, two neighboring states separated by $d\boldsymbol{z}$ acquire the separation

$$d\boldsymbol{z}_{\text{new}} = \big[I + h\mathsf{B} + O(h^2)\big] d\boldsymbol{z}.$$

The common translation of the two states does not change their separation. It is the variation of the velocity from point to point that stretches or shears their small neighborhood.

A linear map changes a small volume by its determinant. The first-order determinant identity is

$$\det(I + h\mathsf{B}) = 1 + h\ \operatorname{tr}\mathsf{B} + O(h^2). \tag{20}$$

To see the linear term, expand the determinant as a sum of permutation products. The identity permutation contributes the sum of diagonal corrections hB_{aa}. Every nonidentity permutation uses at least two off-diagonal entries and therefore first contributes at order h^2. Thus shear can change shape at first order without itself supplying a first-order volume change.

The trace is the divergence of the phase-space field:

$$\operatorname{div} \boldsymbol{u} \equiv \sum_{a=1}^{2N} \frac{\partial u_a}{\partial z_a} = \sum_{i=1}^{N} \left(\frac{\partial \dot{q}_i}{\partial q_i} + \frac{\partial \dot{p}_i}{\partial p_i} \right). \tag{21}$$

It measures the instantaneous *fractional rate of change of an infinitesimal transported volume*, not the speed of a trajectory and not the rate of change of its energy.

8.2 Hamilton's equations make that rate vanish

Substituting (1) into (21),

$$\operatorname{div} \boldsymbol{u} = \sum_i \left(\frac{\partial^2 H}{\partial q_i \partial p_i} - \frac{\partial^2 H}{\partial p_i \partial q_i} \right) = 0. \tag{22}$$

The cancellation uses commuting mixed second derivatives of H, not the assumption $\partial H/\partial t = 0$. The plus sign in the q equation and minus sign in the p equation make the cancellation possible.

We can turn the local short-time statement into a finite-time result. Let

$$\mathsf{U}(t) = \frac{\partial \Phi_{t,t_0}(\boldsymbol{z}_0)}{\partial \boldsymbol{z}_0^T}, \qquad \mathcal{D}(t) = \det \mathsf{U}(t).$$

The matrix U carries an initial infinitesimal parallelepiped to its current one. Composing the flow up to t with the next short step gives

$$\mathcal{D}(t+h) = \big[1 + h \operatorname{div} \boldsymbol{u}(\boldsymbol{z}(t), t) + O(h^2)\big] \mathcal{D}(t).$$

Subtract $\mathcal{D}(t)$, divide by h, and take the limit:

$$\dot{\mathcal{D}} = \big(\operatorname{div} \boldsymbol{u}\big)\mathcal{D}, \qquad \mathcal{D}(t_0) = 1. \tag{23}$$

The Hamiltonian cancellation therefore gives $\mathcal{D}(t) = 1$ exactly, throughout the interval where the flow exists. Consequently the canonical volume measure

$$\boxed{d\Gamma = \prod_{i=1}^{N} dq_i \, dp_i} \tag{24}$$

is preserved. For a region $\mathcal{R}_0$ transported into $\mathcal{R}(t)$, the change-of-variables formula gives

$$\boxed{\int_{\mathcal{R}(t)} d\Gamma = \int_{\mathcal{R}_0} |\mathcal{D}(t)| \, d\Gamma_0 = \int_{\mathcal{R}_0} d\Gamma_0.} \tag{25}$$

This is *Liouville's theorem*. The region's boundary moves with the flow. A fixed box in phase space need not retain the same set of states: states can enter and leave it.

The measure is expressed in canonical variables. The axes have different physical units in general, and the flow need not preserve Euclidean lengths, angles, or the apparent density obtained using arbitrary noncanonical plotting coordinates. "Incompressible" refers specifically to the canonical volume, not to a rigid motion of a phase portrait.

9 Volume preservation in examples

9.1 A free-particle cloud shears without changing area

For $H = p^2/(2m)$, the exact map over time $\tau > 0$ is

$$q = q_0 + \frac{\tau}{m} p_0, \qquad p = p_0.$$

Higher-momentum states move farther horizontally, so a rectangle of initial states becomes a parallelogram. Direct differentiation verifies the area statement:

$$\begin{pmatrix} \partial q/\partial q_0 & \partial q/\partial p_0 \\ \partial p/\partial q_0 & \partial p/\partial p_0 \end{pmatrix} = \begin{pmatrix} 1 & \tau/m \\ 0 & 1 \end{pmatrix}, \qquad \det = 1. \tag{26}$$

Here the displayed matrix is explicitly the component Jacobian with rows (q, p) and columns (q_0, p_0). Its off-diagonal entry tilts the sides while its determinant preserves area.

For the figure, use dimensionless plotting coordinates $\widehat{q} = q/q_*$, $\widehat{p} = p/p_*$, with positive reference scales chosen so that $\tau p_*/(mq_*) = 1$. Then the map is $(\widehat{q}, \widehat{p}) = (\widehat{q}_0 + \widehat{p}_0, \widehat{p}_0)$, with the same scale on corresponding axes in both panels.

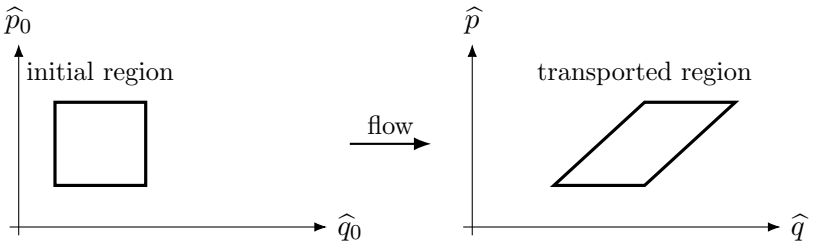

Figure 1: An exact free-particle shear. Both polygons have unit area in the displayed dimensionless coordinates. The cloud changes shape because its states have different velocities, not because states are created or lost. Liouville preservation concerns the area of the cloud; it does not say that the cloud must keep its shape.

An oscillator illustrates a different possibility. Its exact map is

$$\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} \cos\omega\tau & \sin\omega\tau/(m\omega) \\ -m\omega\sin\omega\tau & \cos\omega\tau \end{pmatrix} \begin{pmatrix} q_0 \\ p_0 \end{pmatrix}.$$

Its determinant is $\cos^2\omega\tau + \sin^2\omega\tau = 1$. Energy ellipses are preserved as well, but for a separate reason: this particular Hamiltonian is autonomous. At $q > 0, p = 0$, $\dot{p} < 0$, so the motion in the usual q-horizontal, p-vertical plot is clockwise.

9.2 A changing energy does not imply a changing volume

Return to the prescribed-stiffness oscillator (10). Its equations are

$$\dot{q} = \frac{p}{m}, \qquad \dot{p} = -k(t)q.$$

Its divergence is still

$$\frac{\partial \dot{q}}{\partial q} + \frac{\partial \dot{p}}{\partial p} = 0 + 0 = 0,$$

although $\dot{H} = \frac{1}{2}\dot{k}(t)q^2$ is generally nonzero. Thus

Energy conservation:	a scalar is constant along each trajectory;
Liouville preservation:	a transported collection retains its canonical volume.

The first requires the appropriate conservation condition; the second follows from the canonical Hamilton equations and smoothness. They answer different questions about the same flow.

9.3 Dissipation in a reduced description

For a linearly damped oscillator with physical momentum $p = m\dot{q}$, write

$$\dot{q} = \frac{p}{m}, \qquad \dot{p} = -kq - \gamma p, \qquad \gamma > 0.$$

In these variables the divergence is $-\gamma$. Equation (23) now gives

$$\mathcal{D}(t) = e^{-\gamma(t-t_0)},$$

so a cloud contracts. Its mechanical energy $E = p^2/(2m) + kq^2/2$ obeys

$$\dot{E} = -\frac{\gamma p^2}{m} \leq 0.$$

These are two separate calculations: contraction is measured by the divergence, energy loss by the energy derivative. The dissipative equations above cannot be Hamilton's canonical equations for any smooth $H(q, p, t)$ with this physical pair, because every such Hamiltonian would give zero divergence.

This does not contradict a larger Hamiltonian description including the degrees of freedom that receive the energy. Projecting onto a reduced set of variables need not preserve the full system's volume property. Nor does the statement exclude a reformulation using different, time-dependent canonical momenta; the canonical volume would then be a different measure from the plotted $dq\, d(m\dot{q})$.

In a smooth Hamiltonian flow, an open set of positive canonical volume cannot be squeezed into arbitrarily small-volume neighborhoods of a compact lower-dimensional attractor while remaining bounded there. Individual trajectories can nevertheless approach a saddle along its stable set; that exceptional set need not have positive phase-space volume. Liouville's theorem constrains collections of states, not every possible limiting behavior of one state.

9.4 Exact flow versus a finite-step numerical rule

A short computational caution. The first-order volume calculation used a limit to obtain an exact differential equation. It does not certify an arbitrary finite-step update. For the oscillator, simultaneous forward Euler gives

$$\begin{pmatrix} q_{n+1} \\ p_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & h/m \\ -hm\omega^2 & 1 \end{pmatrix} \begin{pmatrix} q_n \\ p_n \end{pmatrix}, \qquad \det = 1 + \omega^2 h^2.$$

It agrees with the continuous equations to first order but expands area at each nonzero step. The determinant has no linear term, just as zero divergence predicts, yet the quadratic error accumulates. Lecture 15's discrete variational map had determinant one for a structural reason of its own. Exact Hamiltonian flow, a variational discrete map, and a generic numerical approximation must not be silently identified.

10 Compress the structure only after it is visible

With the coordinate order $\boldsymbol{z} = (\boldsymbol{q}^T, \boldsymbol{p}^T)^T$, define the constant block matrix

$$\boxed{\mathsf{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.} \tag{27}$$

Each identity matrix is $N \times N$. The column derivative is

$$\frac{\partial F}{\partial \boldsymbol{z}} = \begin{pmatrix} \partial F/\partial \boldsymbol{q} \\ \partial F/\partial \boldsymbol{p} \end{pmatrix}.$$

Multiplication by J sends $(\partial H/\partial \boldsymbol{q}, \partial H/\partial \boldsymbol{p})$ to $(\partial H/\partial \boldsymbol{p}, -\partial H/\partial \boldsymbol{q})$, exactly the two component equations already established. Hence

$$\boxed{\dot{\boldsymbol{z}} = \mathsf{J}\frac{\partial H}{\partial \boldsymbol{z}}, \qquad \{F, G\} = \left(\frac{\partial F}{\partial \boldsymbol{z}}\right)^T \mathsf{J}\frac{\partial G}{\partial \boldsymbol{z}}.} \tag{28}$$

The antisymmetry $\mathsf{J}^T = -\mathsf{J}$ gives antisymmetry of the bracket by transposing a scalar. The zeros and identity blocks encode the conjugate pairing; they do not define a Euclidean inner product on phase space or make coordinates and momenta have the same units.

This is a compression, not a new premise. The component calculations tell us what the matrix does and why its signs matter. Lecture 17 will ask which changes of phase-space coordinates preserve this pairing. In several degrees of freedom, preserving the full pairing is stronger than preserving only total volume; we have proved the volume result here without replacing the later canonical-transformation criterion by a determinant test.

11 Summary and the next question

The organizing object is the two-function operation $\{F, G\}$. With $G = H$, it gives the change induced by physical motion; with another generator, it gives the change induced by that transformation. The complete time derivative still includes explicit clock dependence:

$$\boxed{\dot{F} = \{F, H\} + \frac{\partial F}{\partial t}, \qquad \delta_G F = \epsilon\{F, G\} \quad \text{at fixed time.}}$$

The examples separate three claims that can otherwise sound identical. Momentum generates translations, whether or not the Hamiltonian is translation invariant. It is conserved when the symmetry and time-dependence conditions make its total derivative vanish. Two conserved quantities need not be in involution with one another.

Liouville's theorem makes a different kind of statement. Hamilton's equations give zero phase-space divergence, divergence gives the fractional rate of change of a transported volume, and the finite-time Jacobian therefore remains one. The result allows shear and changing energy; it concerns canonical volume rather than the shape of a cloud or the value of one observable.

Finally, suppose we describe phase space by new functions $\boldsymbol{Q}(\boldsymbol{q}, \boldsymbol{p})$ and $\boldsymbol{P}(\boldsymbol{q}, \boldsymbol{p})$. Which choices preserve

$$\{Q_i, Q_j\} = 0, \qquad \{P_i, P_j\} = 0, \qquad \{Q_i, P_j\} = \delta_{ij}?$$

The brackets here are computed using the original canonical variables. Preserving them should let the new variables retain the same Hamiltonian structure. Establishing and using that criterion is the job of Lecture 17; generating functions and action–angle variables remain later topics.