## Graduate Classical Mechanics

# Lecture 14: Rigid Bodies II

Dynamics, stability, symmetry, and the heavy top

## 1 The organizing picture

Lecture 13 separated rigid-body configuration into a moving center of mass and a moving orthonormal frame,

$$X(t) = \boldsymbol{R}(t)\mathbf{1}^T + Q(t)X_0,$$

and introduced the body and space descriptions of angular velocity. In particular,

$$\boxed{\dot{Q} = W_s Q = Q W_b, \qquad W_s \boldsymbol{r} = \boldsymbol{\omega}_s \times \boldsymbol{r}, \qquad \boldsymbol{\omega}_b = Q^T \boldsymbol{\omega}_s.}$$

The inertia tensor is constant in body components,

$$\boxed{\boldsymbol{J}_b = \mathsf{I}_b \boldsymbol{\omega}_b,}$$

while ordinary inertial-frame differentiation is simplest in the laboratory frame.

This lecture combines those two advantages. The main required chain is

$$\boxed{\begin{array}{c} \dot{\boldsymbol{J}}_s = \boldsymbol{\tau}_s \quad \Longrightarrow \quad \boldsymbol{\tau}_b = \dot{\boldsymbol{J}}_b + \boldsymbol{\omega}_b \times \boldsymbol{J}_b \\ \Downarrow \\ \mathsf{I}_b \dot{\boldsymbol{\omega}}_b + \boldsymbol{\omega}_b \times (\mathsf{I}_b \boldsymbol{\omega}_b) = \boldsymbol{\tau}_b \\ \Downarrow \\ \text{principal axes} \rightarrow \text{torque-free invariants} \rightarrow \text{stability} \rightarrow \text{symmetric top.} \end{array}}$$

After that required material, we will use the same structural viewpoint to discuss orientation representations and the heavy symmetric top.

# I. Rigid-body dynamics: required through the symmetric top

## 2 Angular-momentum balance in a moving frame

First fix the point about which angular momentum and inertia are measured. For a freely moving rigid body, we use angular momentum of the relative motion about its center of mass; translation of the center of mass is treated separately. For a body with a pivot fixed in an inertial laboratory, we instead use angular momentum and inertia about that pivot. The equations below apply in either of these settings, with torque taken about the same point. They are not a rule for an arbitrary accelerating reference point without additional terms.

In the inertial laboratory frame, angular-momentum balance about the chosen origin is

$$\boxed{\dot{\boldsymbol{J}}_s = \boldsymbol{\tau}_s.}$$

The subscript $s$ means that the vector is written in space/laboratory components.

Lecture 13 established the derivative rule for a physical vector whose body components are related to its space components by

$$\boldsymbol{b}_s = Q\boldsymbol{b}_b :$$

$$\boxed{Q^T\dot{\boldsymbol{b}}_s = \dot{\boldsymbol{b}}_b + \boldsymbol{\omega}_b \times \boldsymbol{b}_b.}$$

Apply this directly to the angular momentum:

$$Q^T\dot{\boldsymbol{J}}_s = \dot{\boldsymbol{J}}_b + \boldsymbol{\omega}_b \times \boldsymbol{J}_b.$$

Since

$$\boldsymbol{\tau}_b = Q^T\boldsymbol{\tau}_s,$$

the laboratory balance law becomes

$$\boxed{\boldsymbol{\tau}_b = \dot{\boldsymbol{J}}_b + \boldsymbol{\omega}_b \times \boldsymbol{J}_b.}$$

This equation is the essential bridge between the two frames:

$$\boxed{\begin{array}{c}\text{space frame: } \dot{\boldsymbol{J}}_s = \boldsymbol{\tau}_s \text{ has the simplest derivative,} \\ \text{body frame: } \mathsf{I}_b \text{ is constant.}\end{array}}$$

The cross-product term is not a new torque. It appears because the basis used for the components is itself rotating.

Using the Lecture 13 relation

$$\boldsymbol{J}_b = \mathsf{I}_b\boldsymbol{\omega}_b,$$

and the fact that $\mathsf{I}_b$ is constant in body components,

$$\dot{\boldsymbol{J}}_b = \mathsf{I}_b\dot{\boldsymbol{\omega}}_b.$$

Therefore

$$\boxed{\mathsf{I}_b\dot{\boldsymbol{\omega}}_b + \boldsymbol{\omega}_b \times (\mathsf{I}_b\boldsymbol{\omega}_b) = \boldsymbol{\tau}_b.}$$

This is Euler's rigid-body equation in vector form.

A useful check is the isotropic case. If

$$\mathsf{I}_b = II_{3\times 3},$$

where $I_{3\times 3}$ denotes the identity matrix, then

$$\boldsymbol{\omega} \times (\mathsf{I}\boldsymbol{\omega}) = I\boldsymbol{\omega} \times \boldsymbol{\omega} = \boldsymbol{0},$$

so Euler's equation reduces to

$$\boxed{I\dot{\boldsymbol{\omega}} = \boldsymbol{\tau}.}$$

For an isotropic body, angular momentum is always parallel to angular velocity, and the nonlinear cross term disappears.

## 3 Principal axes and Euler's component equations

The body inertia tensor is real and symmetric. Therefore it possesses an orthonormal eigenbasis. Choose the body axes to be that eigenbasis. In those *principal axes*,

$$\boxed{\mathsf{I}_b = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}.}$$

The angular momentum is then

$$\boxed{\boldsymbol{J}_b = \begin{pmatrix} I_1\omega_1 \\ I_2\omega_2 \\ I_3\omega_3 \end{pmatrix}.}$$

Substitute this form into the vector equation

$$\mathsf{I}_b\dot{\boldsymbol{\omega}}_b + \boldsymbol{\omega}_b \times (\mathsf{I}_b\boldsymbol{\omega}_b) = \boldsymbol{\tau}_b.$$

Expanding the cross product gives the three Euler equations:

$$\boxed{I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 = \tau_1,}$$

$$\boxed{I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1 = \tau_2,}$$

$$\boxed{I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 = \tau_3.}$$

The use of principal axes is another instance of the symmetric-matrix logic encountered in small oscillations: choose an orthogonal eigenbasis to simplify a quadratic form. Here that choice diagonalizes the relation between $\boldsymbol{J}$ and $\boldsymbol{\omega}$.

However, diagonalizing the inertia tensor does *not* diagonalize the dynamics. The Euler equations remain coupled through products such as $\omega_2\omega_3$. The nonlinear coupling is a dynamical consequence of describing the motion in a rotating basis.

In the torque-free case, one immediate exact solution is a principal-axis rotation. If, for example,

$$\boldsymbol{\omega} = \Omega\boldsymbol{e}_1,$$

then

$$\omega_2 = \omega_3 = 0,$$

and all quadratic coupling terms vanish. The same is true for rotation about either of the other two principal axes.

## 4 Torque-free motion: use invariants before solving

Now set the external torque to zero:

$$\boxed{\boldsymbol{\tau} = \mathbf{0}.}$$

In the inertial frame,

$$\boxed{\dot{\boldsymbol{J}}_s = \mathbf{0}, \qquad \boldsymbol{J}_s = \text{constant.}}$$

The vector $\boldsymbol{J}_s$ is fixed in both magnitude and direction.

Because the body and space components represent the same physical vector,

$$J^2 = \boldsymbol{J} \cdot \boldsymbol{J}$$

is invariant. In principal-axis body components,

$$\boxed{J^2 = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 = \text{constant.}}$$

The rotational kinetic energy is

$$T_{\text{rot}} = \frac{1}{2}\boldsymbol{\omega}_b^T \mathsf{I}_b \boldsymbol{\omega}_b,$$

so in principal axes

$$\boxed{E = T_{\text{rot}} = \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right) = \text{constant.}}$$

Energy conservation follows directly from Euler's vector equation. For zero torque,

$$\mathsf{I}\dot{\boldsymbol{\omega}} = -\boldsymbol{\omega} \times (\mathsf{I}\boldsymbol{\omega}).$$

Therefore

$$\begin{aligned}\frac{dT}{dt} &= \boldsymbol{\omega} \cdot \mathsf{I}\dot{\boldsymbol{\omega}} \\ &= -\boldsymbol{\omega} \cdot [\boldsymbol{\omega} \times (\mathsf{I}\boldsymbol{\omega})] \\ &= 0.\end{aligned}$$

Thus

$$\boxed{\frac{dT}{dt} = 0.}$$

In the three-dimensional space of body components $(\omega_1, \omega_2, \omega_3)$, the two conservation laws define two quadratic surfaces:

$$I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 = 2E,$$
$$I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 = J^2.$$

The torque-free trajectory must lie on their intersection. Even before solving the Euler equations explicitly, the invariants strongly constrain the allowed motion.

## 5 Stability of principal-axis rotation

Assume three distinct principal moments ordered as

$$\boxed{I_1 < I_2 < I_3.}$$

Each principal-axis rotation is an exact torque-free solution. The question is whether a small perturbation away from that solution remains small.

This is exactly the stability strategy used in Lecture 12: choose an exact solution, perturb it, linearize the equations for the perturbation, and inspect the sign of the resulting second-order equation.

### 5.1 Rotation about the smallest principal moment

Take a steady rotation about axis 1 and add small transverse components:

$$\omega_1 \simeq \Omega, \qquad \omega_2 = \eta_2, \qquad \omega_3 = \eta_3,$$

with $\eta_2$ and $\eta_3$ small.

The torque-free Euler equations are

$$I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 = 0,$$
$$I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1 = 0,$$
$$I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 = 0.$$

To first order in the perturbations, the first equation gives no change in the leading constant $\Omega$, while the other two give

$$\boxed{\dot{\eta}_2 = \frac{I_3 - I_1}{I_2}\Omega\eta_3,}$$

$$\boxed{\dot{\eta}_3 = -\frac{I_2 - I_1}{I_3}\Omega\eta_2.}$$

Differentiate the first equation once more and eliminate $\eta_3$:

$$\boxed{\ddot{\eta}_2 = -\Omega^2\frac{(I_3 - I_1)(I_2 - I_1)}{I_2 I_3}\eta_2.}$$

Every factor multiplying the minus sign is positive, so the perturbation equation is oscillatory. Rotation about axis 1 is linearly stable.

### 5.2 Rotation about the intermediate principal moment

Now rotate about axis 2:

$$\omega_2 \simeq \Omega, \qquad \omega_1 = \eta_1, \qquad \omega_3 = \eta_3.$$

Retain only terms containing one transverse perturbation. The first and third Euler equations give

$$\dot{\eta}_1 = -\frac{I_3 - I_2}{I_1}\Omega\eta_3, \qquad \dot{\eta}_3 = -\frac{I_2 - I_1}{I_3}\Omega\eta_1.$$

The two coupling coefficients now have the same sign. Differentiating the first equation and using the second gives

$$\boxed{\ddot{\eta}_1 = +\Omega^2\frac{(I_3 - I_2)(I_2 - I_1)}{I_1 I_3}\eta_1.}$$

Define the positive growth rate

$$\lambda = |\Omega|\sqrt{\frac{(I_3 - I_2)(I_2 - I_1)}{I_1 I_3}}.$$

Then $\eta_1 = Ae^{\lambda t} + Be^{-\lambda t}$. Special initial perturbations can select the decaying branch, but a generic perturbation has a growing component. That is enough to make rotation about axis 2 linearly unstable. Once the transverse components become comparable to $|\Omega|$, the linearization no longer describes the subsequent motion; it predicts the initial loss of stability, not the full nonlinear flip.

### 5.3 Rotation about the largest principal moment

Set $\omega_3 \simeq \Omega$, $\omega_1 = \eta_1$, and $\omega_2 = \eta_2$. The first two Euler equations become

$$\dot{\eta}_1 = -\frac{I_3 - I_2}{I_1}\Omega\eta_2, \qquad \dot{\eta}_2 = \frac{I_3 - I_1}{I_2}\Omega\eta_1.$$

Consequently,

$$\boxed{\ddot{\eta}_1 = -\Omega^2 \frac{(I_3 - I_2)(I_3 - I_1)}{I_1 I_2}\eta_1.}$$

The coefficient is negative, as it was for axis 1, so both transverse components remain oscillatory in the linearized problem. Axis 3 is linearly stable.

Here "stable" means that sufficiently small transverse perturbations remain bounded in this linear analysis, not that they decay. There is no damping to bring the body back to its original rotation. A small longitudinal perturbation merely changes $\Omega$ to first order. The test concerns the angular-velocity components near a steady rotation; it does not assert that two orientations with slightly different spin rates remain phase-aligned forever. The distinct-moment assumption matters: when moments coincide, a vanishing coefficient calls for the corresponding symmetric problem rather than an exponential-versus-oscillatory verdict from these formulas.

The three cases can be summarized as

$$\boxed{\begin{array}{c} I_1 < I_2 < I_3, \\ \text{rotation about } I_1 : \text{ stable}, \\ \text{rotation about } I_2 : \text{ unstable}, \\ \text{rotation about } I_3 : \text{ stable}. \end{array}}$$

Equivalently,

$$\boxed{\text{smallest and largest principal moments are stable; the intermediate one is unstable.}}$$

This is the intermediate-axis instability. A familiar qualitative demonstration is the toss of a rectangular object: rotations about the principal axes associated with the smallest and largest moments are robust, while rotation about the intermediate-moment axis flips under small perturbations.

## 6 Torque-free symmetric top

A particularly important simplification occurs when two principal moments are equal:

$$\boxed{I_1 = I_2 = I_\perp, \qquad I_3 = I_\parallel.}$$

The third axis is the body symmetry axis.

With zero torque, the Euler equations become

$$\begin{aligned} I_\perp\dot{\omega}_1 + (I_\parallel - I_\perp)\omega_2\omega_3 &= 0, \\ I_\perp\dot{\omega}_2 + (I_\perp - I_\parallel)\omega_3\omega_1 &= 0, \\ I_\parallel\dot{\omega}_3 &= 0. \end{aligned}$$

Thus

$$\boxed{\dot{\omega}_3 = 0.}$$

The component of angular velocity along the symmetry axis is constant.

Define

$$\boxed{\Omega_b = \frac{I_\parallel - I_\perp}{I_\perp}\,\omega_3.}$$

Then the transverse equations are

$$\boxed{\dot{\omega}_1 = -\Omega_b \omega_2, \qquad \dot{\omega}_2 = \Omega_b \omega_1.}$$

These are uniform rotation equations in the $(\omega_1, \omega_2)$ plane. Equivalently,

$$\boxed{\omega_1 + i\omega_2 = Ce^{i\Omega_b t}.}$$

So in body components, the transverse part of angular velocity rotates uniformly around the body symmetry axis. With initial components $a = \omega_1(0)$ and $b = \omega_2(0)$, the same solution is

$$\omega_1(t) = a\cos(\Omega_b t) - b\sin(\Omega_b t), \qquad \omega_2(t) = a\sin(\Omega_b t) + b\cos(\Omega_b t).$$

Thus $\omega_1^2 + \omega_2^2 = a^2 + b^2$: this is motion of components in the body basis, with signed rate $\Omega_b$, not yet the rate at which a visible axis moves in the laboratory.

The body-frame picture is not the whole physical picture. In space, because the motion is torque free,

$$\boxed{\boldsymbol{J}_s = \text{constant.}}$$

To connect the two descriptions, let

$$\boxed{\boldsymbol{n} = Q\boldsymbol{e}_3}$$

be the body symmetry axis written in laboratory components. The rigid-body kinematics of Lecture 13 gives

$$\boxed{\dot{\boldsymbol{n}} = \boldsymbol{\omega}_s \times \boldsymbol{n}.}$$

For a symmetric body, angular momentum can be written directly in terms of $\boldsymbol{\omega}_s$ and the symmetry axis:

$$\boxed{\boldsymbol{J}_s = I_\perp \boldsymbol{\omega}_s + (I_\parallel - I_\perp)(\boldsymbol{\omega}_s \cdot \boldsymbol{n})\boldsymbol{n}.}$$

Cross this equation with $\boldsymbol{n}$. The second term vanishes because $\boldsymbol{n} \times \boldsymbol{n} = 0$:

$$\boldsymbol{J}_s \times \boldsymbol{n} = I_\perp \boldsymbol{\omega}_s \times \boldsymbol{n}.$$

Using $\dot{\boldsymbol{n}} = \boldsymbol{\omega}_s \times \boldsymbol{n}$,

$$\boxed{\dot{\boldsymbol{n}} = \frac{1}{I_\perp}\boldsymbol{J}_s \times \boldsymbol{n}.}$$

This equation has a direct geometric interpretation. The vector $\boldsymbol{J}_s$ is fixed in the laboratory frame, and $\dot{\boldsymbol{n}}$ is perpendicular to both $\boldsymbol{J}_s$ and $\boldsymbol{n}$. Therefore the body symmetry axis precesses uniformly about the fixed angular momentum vector.

The space-frame precession rate is

$$\boxed{\Omega_s = \frac{J}{I_\perp}.}$$

For $J > 0$, the cone angle $\beta$ between $\boldsymbol{n}$ and $\boldsymbol{J}_s$ is constant because

$$\frac{d}{dt}(\boldsymbol{J}_s \cdot \boldsymbol{n}) = \frac{1}{I_\perp}\boldsymbol{J}_s \cdot (\boldsymbol{J}_s \times \boldsymbol{n}) = 0, \qquad \cos\beta = \frac{I_\parallel \omega_3}{J}.$$

One can choose the space $z$ axis along $\boldsymbol{J}_s$. Then $\dot{n}_x = -\Omega_s n_y$, $\dot{n}_y = \Omega_s n_x$, and $\dot{n}_z = 0$, making the uniform circular precession explicit. If $\boldsymbol{n}$ is parallel to $\boldsymbol{J}_s$, the cone collapses and $\dot{\boldsymbol{n}} = 0$; there is no visible motion of the symmetry axis even though the body may spin.

The two rates $\Omega_b$ and $\Omega_s$ answer different questions and need not agree. For instance, in the isotropic limit $I_\parallel = I_\perp$, the body components of $\boldsymbol{\omega}$ are constant, while a marked body axis not aligned with the fixed $\boldsymbol{J}_s$ can still rotate around it in space.

Thus the torque-free symmetric top has two complementary descriptions:

body frame: transverse components of $\boldsymbol{\omega}_b$ rotate about the symmetry axis,
space frame: the symmetry axis $\boldsymbol{n}$ precesses about fixed $\boldsymbol{J}_s$.

**Exam II rigid-body boundary.**

Required rigid-body material ends here: Lecture 13 kinematics together with Euler's equations, principal axes, torque-free invariants, intermediate-axis stability, and the torque-free symmetric top. The remainder of these notes is conceptual extension and is not part of Exam II.

# II. Conceptual extension beyond the Exam II boundary

Everything in Part II, including its worked checks and reconstruction formulas, is optional enrichment outside Exam II. We use the heavy top to see how the same mechanics can be simplified by choosing a description adapted to symmetry; we are not adding another required rigid-body syllabus.

## 7 Orientation representations: choose a description for a purpose

The physical orientation is still the frame map

$$\boxed{Q \in SO(3).}$$

There are several useful ways to describe the same orientation. Each emphasizes different structure.

| **Description** | **Advantage** | **Tradeoff** |
|---|---|---|
| $Q$ | Direct frame map; acts immediately on points and axes; globally nonsingular | Nine entries constrained by orthogonality |
| Euler coordinates | Three local scalar generalized coordinates; cyclic coordinates can make reduction transparent | Convention dependent; coordinate singularities; angle rates are not angular-velocity components |
| Rotor $\mathcal{R}$ | Compact geometric composition; no coordinate singularity; natural generator/bivector relation | Unit constraint, double-cover sign, and less familiar notation |

Euler angles should therefore not be interpreted as the physical definition of orientation. They are one local coordinate choice or one factorization of the same matrix $Q$. Their usefulness appears when scalar generalized coordinates expose a symmetry or produce a cyclic variable.

A rotor gives another representation of the same physical rotation. In the notation used earlier,

$$\boxed{\boldsymbol{a}_s = \mathcal{R}\boldsymbol{a}_b\widetilde{\mathcal{R}}, \qquad \mathcal{R}\widetilde{\mathcal{R}} = 1, \qquad \mathcal{R} \sim -\mathcal{R}.}$$

The two signs represent the same physical orientation. We will not redevelop the rotor formalism here; the point is simply that orientation has several representations, and the best one depends on the problem being solved.

## 8 Symmetry suggests how to factor the orientation

For a symmetric top, not all directions attached to the body are physically equivalent. The distinguished direction is the symmetry axis

$$\boxed{\boldsymbol{n} = Q\boldsymbol{e}_3.}$$

This suggests decomposing the orientation into two pieces:

$$\boxed{Q = Q_{\boldsymbol{n}} R_3(\chi),}$$

where

$$\boxed{Q_{\boldsymbol{n}} \boldsymbol{e}_3 = \boldsymbol{n}, \qquad R_3(\chi)\boldsymbol{e}_3 = \boldsymbol{e}_3.}$$

The first factor carries the body symmetry axis into its physical direction $\boldsymbol{n}$. The second factor rotates about that symmetry axis.

Thus

$$\boxed{\text{orientation} = \text{axis direction} + \text{rotation about the axis.}}$$

This is a symmetry-adapted decomposition: the variables are chosen to match the structure of the body.

The decomposition is not unique. For any angle $\alpha$,

$$Q_{\boldsymbol{n}} \to Q_{\boldsymbol{n}} R_3(\alpha), \qquad \chi \to \chi - \alpha,$$

leaves the product $Q$ unchanged. Only the product is physical.

A familiar Euler-type decomposition appears as one local choice of $Q_{\boldsymbol{n}}$. For example, one may take

$$\boxed{Q_{\boldsymbol{n}} = R_z(\phi) R_y(\theta),}$$

so that

$$\boxed{Q = R_z(\phi) R_y(\theta) R_3(\chi).}$$

The important logical order is that the symmetry-adapted variables are identified first; the angular coordinates are introduced afterward as one way to parameterize them.

# 9 The heavy symmetric top with a fixed pivot

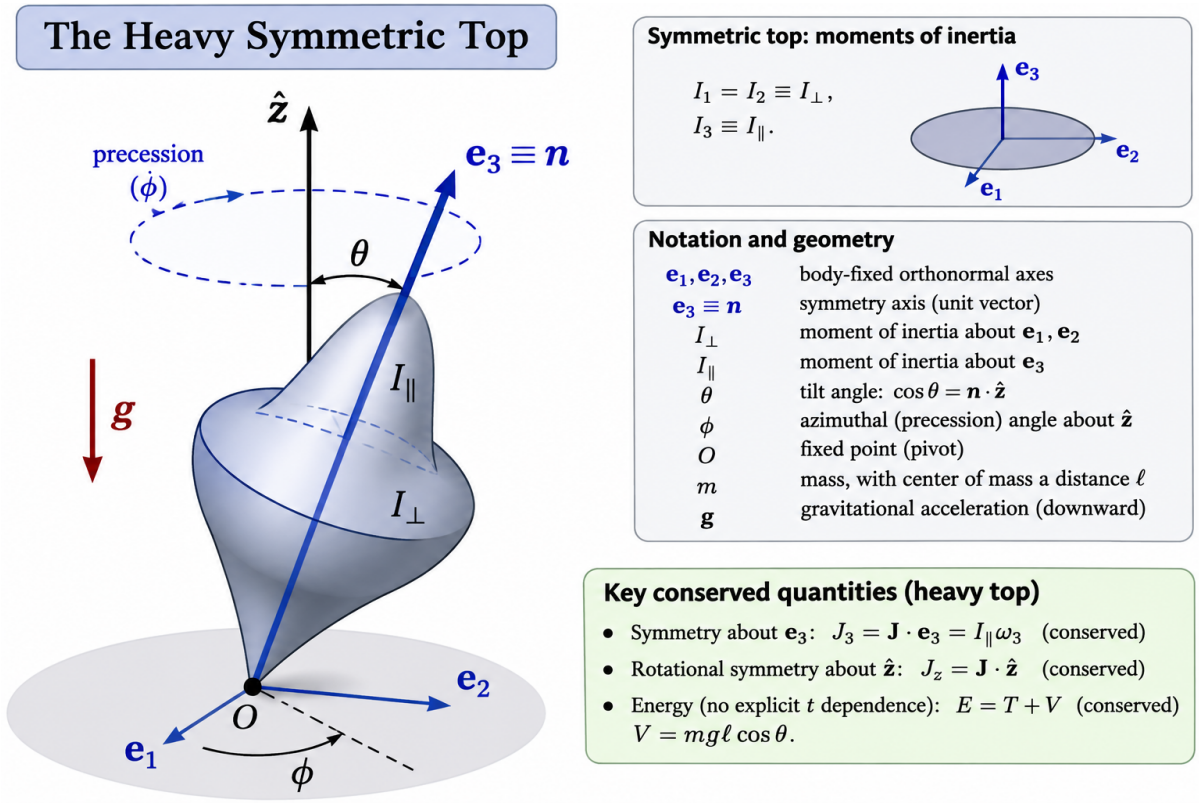


Figure 1: Geometry and notation for the heavy symmetric top. The body symmetry axis is $\boldsymbol{n} = Q\boldsymbol{e}_3$; the principal moments satisfy $I_1 = I_2 = I_\perp$ and $I_3 = I_\parallel$. The tilt $\theta$ and azimuth $\phi$ describe the direction of $\boldsymbol{n}$ relative to the fixed vertical, while $J_3$ and $J_z$ are the conserved angular-momentum components associated with the two rotational symmetries.

Figure 1 collects the geometry before we reduce the dynamics. In the illustration, $\boldsymbol{e}_3 \equiv \boldsymbol{n}$ labels the same geometric axis; our component convention remains $\boldsymbol{n} = Q\boldsymbol{e}_3$, not equality of its body and space component columns. The figure's mass $m$ is denoted by $M$ in the equations below. The distinction between the body-fixed symmetry direction $\boldsymbol{e}_3$ and its laboratory representation $\boldsymbol{n} = Q\boldsymbol{e}_3$ is especially important: $\boldsymbol{e}_3$ is fixed in body coordinates, while $\boldsymbol{n}$ moves in space.

Now let the top have a fixed pivot. Suppose the center of mass lies a distance $\ell$ along the body symmetry axis. Then in space components

$$\boxed{\boldsymbol{n} = Q\boldsymbol{e}_3, \qquad \boldsymbol{r}_{\mathrm{CM}} = \ell\boldsymbol{n}.}$$

Take $\boldsymbol{e}_z$ to point upward. The gravitational potential is

$$\boxed{V = Mg\ell\, \boldsymbol{e}_z \cdot \boldsymbol{n}.}$$

The sign is fixed by this convention: the potential is largest when the center of mass is highest along $+\boldsymbol{e}_z$.

Use principal moments about the pivot,

$$\boxed{\mathsf{I}_O = \mathrm{diag}(I_\perp, I_\perp, I_\parallel).}$$

If the inertia tensor about the center of mass is known instead, the pivot tensor may be obtained from the parallel-axis relation

$$\mathsf{I}_O = \mathsf{I}_{\mathrm{CM}} + M\ell^2(I_{3\times 3} - \boldsymbol{e}_3\boldsymbol{e}_3^T).$$

Both tensors in this equation are expressed in the same body axes. The added term increases each transverse moment by $M\ell^2$ but leaves the axial moment unchanged, because shifting the reference point along the symmetry axis does not change a mass element's distance from that axis. With pivot moments, $T$ already includes the center of mass moving around the pivot; adding a separate $\frac{1}{2}M|\dot{\boldsymbol{r}}_{\mathrm{CM}}|^2$ to that pivot-based rotational energy would count part of the motion twice.

The pivot force has zero moment about the pivot. Gravity therefore supplies the torque

$$\boxed{\boldsymbol{\tau} = (\ell\boldsymbol{n}) \times (-Mg\boldsymbol{e}_z) = -Mg\ell\,\boldsymbol{n} \times \boldsymbol{e}_z.} \tag{1}$$

From this point on, un-subscripted vectors $\boldsymbol{n}, \boldsymbol{\omega}, \boldsymbol{J}, \boldsymbol{\tau}$ are in laboratory components, and all moments of inertia refer to the pivot.

## 10 Decompose angular velocity without choosing three Euler angles

The symmetry axis satisfies

$$\dot{\boldsymbol{n}} = \boldsymbol{\omega} \times \boldsymbol{n}.$$

Since $\boldsymbol{n} \cdot \boldsymbol{n} = 1$, differentiation gives $\boldsymbol{n} \cdot \dot{\boldsymbol{n}} = 0$: the change of the axis is tangent to its unit sphere. Cross the kinematic relation with $\boldsymbol{n}$ and use the vector triple-product identity:

$$\boldsymbol{n} \times \dot{\boldsymbol{n}} = \boldsymbol{n} \times (\boldsymbol{\omega} \times \boldsymbol{n}) = \boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \boldsymbol{n})\boldsymbol{n}.$$

Thus the motion of $\boldsymbol{n}$ determines only the component of $\boldsymbol{\omega}$ perpendicular to $\boldsymbol{n}$. A rotation about $\boldsymbol{n}$ does not move that axis, so it cannot be inferred from $\dot{\boldsymbol{n}}$ alone. The full decomposition is

$$\boxed{\boldsymbol{\omega} = \boldsymbol{n} \times \dot{\boldsymbol{n}} + \sigma\boldsymbol{n}, \qquad \sigma = \boldsymbol{\omega} \cdot \boldsymbol{n}.}$$

The first term is the angular velocity perpendicular to the symmetry axis required to move that axis. The scalar $\sigma$ is the *physical axial spin component*. It must not yet be identified with $\dot{\chi}$: the chosen reference frame $Q_{\boldsymbol{n}}$ can itself rotate about $\boldsymbol{n}$ as the axis moves. We will calculate that contribution in the reconstruction step.

For a symmetric top the transverse plane has a single moment $I_\perp$. The two components of $\boldsymbol{\omega}$ in this decomposition are perpendicular, and $|\boldsymbol{n} \times \dot{\boldsymbol{n}}|^2 = |\dot{\boldsymbol{n}}|^2$. Hence the usual quadratic kinetic energy separates as

$$\boxed{T = \frac{1}{2}I_\perp|\dot{\boldsymbol{n}}|^2 + \frac{1}{2}I_\parallel\sigma^2.}$$

Rotation about the body symmetry axis gives a conserved angular-momentum component

$$\boxed{J_3 = I_\parallel\sigma = \text{constant}.}$$

Rotational symmetry about the laboratory vertical gives a second conserved component

$$\boxed{J_z = \boldsymbol{e}_z \cdot \boldsymbol{J} = \text{constant.}}$$

The angular momentum can be written as

$$\boxed{\boldsymbol{J} = I_\perp(\boldsymbol{n} \times \dot{\boldsymbol{n}}) + J_3\boldsymbol{n}.}$$

These statements can also be checked directly, without relying on a coordinate convention for Euler angles. Equation (1) gives

$$\dot{J}_z = \boldsymbol{e}_z \cdot \dot{\boldsymbol{J}} = \boldsymbol{e}_z \cdot \boldsymbol{\tau} = 0.$$

The axial projection needs slightly more care because $\boldsymbol{n}$ moves. From $J_3 = \boldsymbol{J} \cdot \boldsymbol{n}$,

$$\begin{aligned}\frac{d}{dt}(\boldsymbol{J} \cdot \boldsymbol{n}) &= \boldsymbol{\tau} \cdot \boldsymbol{n} + \boldsymbol{J} \cdot \dot{\boldsymbol{n}} \\ &= 0 + \left[I_\perp(\boldsymbol{n} \times \dot{\boldsymbol{n}}) + J_3\boldsymbol{n}\right] \cdot \dot{\boldsymbol{n}} = 0.\end{aligned}$$

Both scalar products vanish for geometric reasons. In particular, zero torque along a *moving* direction is not by itself a general proof that a projection is conserved; the second term is essential here. For an asymmetric body, the displayed form of $\boldsymbol{J}$ would not hold.

The full vector $\boldsymbol{J}$ is generally not constant: gravity changes its horizontal components. Nor are $J_3$ and $J_z$ equal in general, because they project the same vector on different axes. Energy is also conserved for the ideal fixed-pivot problem. Indeed, the rotational power $\boldsymbol{\omega} \cdot \boldsymbol{\tau} = -Mg\ell\, \boldsymbol{e}_z \cdot \dot{\boldsymbol{n}}$ cancels $\dot{V} = Mg\ell\, \boldsymbol{e}_z \cdot \dot{\boldsymbol{n}}$.

These two conserved projections and the energy signal a strong reduction. Instead of introducing three generic orientation coordinates at the start, we can describe the direction $\boldsymbol{n}$ on the unit sphere and retain the physical axial spin through the constant $J_3$.

## 11 Coordinates on the sphere of symmetry-axis directions

Parameterize the unit vector $\boldsymbol{n}$ by the usual polar angles:

$$\boxed{\boldsymbol{n} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}.}$$

These are coordinates on the sphere of possible axis directions, not yet a full three-angle parameterization of the orientation.

Work first on the patch $0 < \theta < \pi$, where the azimuth is defined. The two unit tangent directions are

$$\boldsymbol{e}_\theta = \begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{pmatrix}, \qquad \boldsymbol{e}_\phi = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}.$$

Direct differentiation of the components of $\boldsymbol{n}$ gives

$$\dot{\boldsymbol{n}} = \dot{\theta}\, \boldsymbol{e}_\theta + \sin\theta\, \dot{\phi}\, \boldsymbol{e}_\phi.$$

Because $\boldsymbol{e}_\theta$ and $\boldsymbol{e}_\phi$ are orthogonal unit vectors,

$$\boxed{|\dot{\boldsymbol{n}}|^2 = \dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2.}$$

The factor $\sin\theta$ is geometric: at a fixed latitude, a given change of azimuth travels a shorter distance on the sphere than it does at the equator.

The oriented tangent relations $\boldsymbol{n} \times \boldsymbol{e}_\theta = \boldsymbol{e}_\phi$ and $\boldsymbol{n} \times \boldsymbol{e}_\phi = -\boldsymbol{e}_\theta$ similarly give

$$\boldsymbol{n} \times \dot{\boldsymbol{n}} = \dot{\theta}\,\boldsymbol{e}_\phi - \sin\theta\,\dot{\phi}\,\boldsymbol{e}_\theta.$$

Now $\boldsymbol{e}_z \cdot \boldsymbol{e}_\phi = 0$ and $\boldsymbol{e}_z \cdot \boldsymbol{e}_\theta = -\sin\theta$, so

$$\boxed{(\boldsymbol{n} \times \dot{\boldsymbol{n}}) \cdot \boldsymbol{e}_z = \sin^2\theta\,\dot{\phi}.}$$

Only azimuthal motion of the axis contributes to this vertical projection; changing the tilt alone does not. Therefore the conserved vertical angular momentum becomes

$$\boxed{J_z = I_\perp \sin^2\theta\,\dot{\phi} + J_3\cos\theta.}$$

Solving for the azimuthal rate,

$$\boxed{\dot{\phi} = \frac{J_z - J_3\cos\theta}{I_\perp \sin^2\theta}.}$$

The total energy is

$$E = \frac{1}{2}I_\perp\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) + \frac{J_3^2}{2I_\parallel} + Mg\ell\cos\theta.$$

Fix the conserved values $J_3$ and $J_z$ for the motion under study. Substituting the azimuthal rate into its kinetic-energy contribution gives

$$\frac{1}{2}I_\perp \sin^2\theta\,\dot{\phi}^2 = \frac{(J_z - J_3\cos\theta)^2}{2I_\perp \sin^2\theta}.$$

The axial spin energy $J_3^2/(2I_\parallel)$ is constant, so it can be transferred to the other side of the energy equation. The result is a one-dimensional equation for $\theta$:

$$\boxed{\frac{1}{2}I_\perp\dot{\theta}^2 + V_{\text{eff}}(\theta) = E - \frac{J_3^2}{2I_\parallel},}$$

with

$$\boxed{V_{\text{eff}}(\theta) = Mg\ell\cos\theta + \frac{(J_z - J_3\cos\theta)^2}{2I_\perp \sin^2\theta}.}$$

The effective potential is a function of $\theta$ *at fixed* $J_z$ *and* $J_3$. Its second term is not a new gravitational potential or an added external force. It is the azimuthal kinetic energy expressed using the conserved projections. This is an energy reduction, not a prescription to substitute a conserved momentum into a Lagrangian and vary it as though nothing else changed.

The reduction has the same broad structure seen earlier in central-force mechanics:

$$\boxed{\begin{array}{c}\text{symmetry} \to \text{conserved momentum} \to \text{eliminate a velocity} \\ \Downarrow \\ \text{effective one-dimensional dynamics}\end{array}}$$

Here the reduced coordinate is the nutation angle $\theta$. We still need to say how the energy equation determines its motion.

## 12 Use the reduced energy: nutation and turning points

For brevity define the reduced energy

$$\mathcal{E} = E - \frac{J_3^2}{2I_\parallel}.$$

Then

$$\boxed{\frac{1}{2}I_\perp \dot{\theta}^2 = \mathcal{E} - V_{\text{eff}}(\theta).} \tag{2}$$

The right-hand side must be nonnegative. Thus, at the chosen constants of motion, the allowed tilts are the intervals where $V_{\text{eff}}(\theta) \le \mathcal{E}$. Initial conditions select an allowed interval and an initial direction of motion; the energy equation alone does not choose between increasing and decreasing $\theta$.

On a segment with $\dot{\theta} \neq 0$, let $s = \operatorname{sgn}\dot{\theta}$, held fixed until a turning point. Separating variables gives the actual *quadrature*, meaning an integral that determines time from position:

$$\boxed{t - t_0 = s\sqrt{\frac{I_\perp}{2}} \int_{\theta_0}^{\theta(t)} \frac{d\vartheta}{\sqrt{\mathcal{E} - V_{\text{eff}}(\vartheta)}}.} \tag{3}$$

For a decreasing branch both $s$ and the oriented integral are negative, so elapsed time is positive. Invert this relation, analytically or numerically, to obtain $\theta(t)$. The integral may be more useful than a complicated explicit formula: it displays both the allowed range and the time spent at each tilt.

### 12.1 An ordinary turning point is not a steady top

At a simple turning point $\theta_*$ in this coordinate patch,

$$V_{\text{eff}}(\theta_*) = \mathcal{E}, \qquad \frac{dV_{\text{eff}}}{d\theta}(\theta_*) \neq 0.$$

The axis stops changing tilt momentarily, but the reduced force is nonzero. Differentiating (2) where $\dot{\theta} \neq 0$ gives

$$I_\perp \ddot{\theta} = -\frac{dV_{\text{eff}}}{d\theta}.$$

There is also a check that does not divide by $\dot{\theta}$. From the tangent-basis expression for $\boldsymbol{J}$, use $\dot{\boldsymbol{e}}_\theta \cdot \boldsymbol{e}_\phi = \dot{\phi}\cos\theta$ and $\dot{\boldsymbol{n}} \cdot \boldsymbol{e}_\phi = \dot{\phi}\sin\theta$. Projection of $\dot{\boldsymbol{J}} = \boldsymbol{\tau}$ onto $\boldsymbol{e}_\phi$ gives

$$I_\perp \ddot{\theta} - I_\perp \sin\theta\cos\theta\,\dot{\phi}^2 + J_3 \sin\theta\,\dot{\phi} = Mg\ell\sin\theta.$$

Substituting the conserved-momentum expression for $\dot{\phi}$ recovers $I_\perp\ddot{\theta} = -dV_{\text{eff}}/d\theta$ throughout the patch, including turning points and steady tilts. This is why a root of the energy equation alone cannot be declared a constant-tilt solution. At a simple turning point the axis reverses its tilt motion, and one continues the quadrature with the opposite sign of $s$. A simple zero produces an integrand proportional to $|\vartheta - \theta_*|^{-1/2}$, which is integrable: the turning point is reached in finite time.

If an allowed interval has two simple interior endpoints $\theta_- < \theta_+$, the time for one complete nutation is

$$\boxed{T_{\text{nut}} = \sqrt{2I_\perp} \int_{\theta_-}^{\theta_+} \frac{d\theta}{\sqrt{\mathcal{E} - V_{\text{eff}}(\theta)}}.}$$

This is a period of the *tilt*. The azimuth and residual spin need not return to their initial values after one nutation, so the full orientation need not be periodic with this period.

A constant tilt instead requires both

$$\mathcal{E} = V_{\text{eff}}(\theta_0), \qquad \frac{dV_{\text{eff}}}{d\theta}(\theta_0) = 0.$$

These are the conditions for steady precession at an interior tilt. A nondegenerate minimum supports small tilt oscillations, with squared frequency $(d^2V_{\text{eff}}/d\theta^2)(\theta_0)/I_\perp$. An orbit at the energy of a nondegenerate maximum can approach the corresponding unstable steady motion only asymptotically: the quadrature has a logarithmic divergence there. Thus a repeated root cannot be treated as an ordinary finite-time bounce.

## 12.2 Two checks on the reduced description

**A spinning top released with its axis instantaneously at rest.** Take $0 < \theta_0 < \pi$ and $\dot\theta(0) = \dot\phi(0) = 0$, but allow nonzero axial spin. The initial data give

$$J_z = J_3 \cos\theta_0, \qquad \mathcal{E} = Mg\ell \cos\theta_0.$$

At that point the numerator of the effective kinetic term vanishes, so its first derivative vanishes too. Therefore

$$\frac{dV_{\text{eff}}}{d\theta}(\theta_0) = -Mg\ell \sin\theta_0, \qquad \ddot\theta(0) = \frac{Mg\ell}{I_\perp}\sin\theta_0 > 0.$$

The axis initially tilts farther from the upward vertical. Nonzero spin alone does not make this tilted state steady. As the tilt changes, the conserved $J_z$ forces an azimuthal rate to develop. The initial acceleration can also be obtained directly from the vector torque equation; it does not follow from assigning a constant solution to the first-order square-root equation at its endpoint.

**A horizontal axis in steady precession.** For contrast, let $\theta_0 = \pi/2$ and write the constant azimuthal rate as $\nu$. To check force balance generally, set $B = J_z - J_3\cos\theta$. Differentiation at fixed momenta gives

$$\frac{dV_{\text{eff}}}{d\theta} = -Mg\ell\sin\theta + \frac{J_3 B}{I_\perp \sin\theta} - \frac{B^2\cos\theta}{I_\perp \sin^3\theta}.$$

At a constant tilt, $B = I_\perp \nu \sin^2\theta_0$, and stationarity becomes

$$I_\perp \nu^2 \cos\theta_0 - J_3\nu + Mg\ell = 0.$$

At the equator this reduces to

$$\boxed{\nu = \frac{Mg\ell}{J_3}, \qquad J_z = I_\perp \nu, \qquad J_3 \neq 0.}$$

Choose $\dot\theta = 0$ and $\mathcal{E} = J_z^2/(2I_\perp)$ to obtain this particular steady solution. It differs from release with a stationary axis: the precession is already present. As a direct check, $\boldsymbol{J} = I_\perp \nu \boldsymbol{e}_z + J_3 \boldsymbol{n}$, so

$$\dot{\boldsymbol{J}} = J_3 \nu\, \boldsymbol{e}_z \times \boldsymbol{n} = Mg\ell\, \boldsymbol{e}_z \times \boldsymbol{n} = \boldsymbol{\tau}.$$

The formula is a concrete check of the reduction, not an assumption that every rapidly spinning top is in steady precession.

### 12.3 What happens at a pole?

The formulas containing $1/\sin^2\theta$ were derived on $0 < \theta < \pi$. If $J_z \neq J_3$, then as $\theta \to 0$ the effective kinetic term behaves as

$$\frac{(J_z - J_3)^2}{2I_\perp \theta^2},$$

which excludes the north pole at finite energy. Similarly, the south pole requires $J_z = -J_3$ rather than $J_z = J_3$. These conditions have a direct vector meaning: at $\boldsymbol{n} = \boldsymbol{e}_z$ the two projections of $\boldsymbol{J}$ must agree, and at $\boldsymbol{n} = -\boldsymbol{e}_z$ they must be opposite.

When $J_z = J_3$, the apparent north-pole divergence cancels:

$$\frac{(J_z - J_3\cos\theta)^2}{2I_\perp \sin^2\theta} = \frac{J_3^2}{2I_\perp}\tan^2\left(\frac{\theta}{2}\right),$$

which tends to zero. Thus the pole is not always a physical infinite-energy barrier. Compatibility of the momenta is necessary for reaching it, not sufficient: the energy and initial conditions still matter.

Even in this compatible case, $\phi$ is undefined at the pole. Do not substitute $\theta = 0$ or $\pi$ into an unqualified azimuthal formula or interpret a coordinate jump as a jump in the body. The vector $\boldsymbol{n}$ and orientation $Q$ remain the physical objects; continue them directly or use a different local coordinate patch. An exactly vertical spinning solution is best treated in that vector description.

## 13 Euler angles reappear only for reconstruction

The reduced problem determines the moving axis and its physical axial spin, but not yet the orientation of a marked transverse body direction. To recover it, choose the local reference frame

$$Q_{\boldsymbol{n}} = R_z(\phi)R_y(\theta), \qquad Q = Q_{\boldsymbol{n}}R_3(\chi).$$

Why is $\dot\chi$ not simply $\sigma$? Differentiate the reference-frame product and multiply by $Q_{\boldsymbol{n}}^T$:

$$\dot Q_{\boldsymbol{n}} Q_{\boldsymbol{n}}^T = \dot R_z R_z^T + R_z(\dot R_y R_y^T)R_z^T.$$

The first term is rotation at rate $\dot\phi$ about the space vertical. The second is rotation at rate $\dot\theta$ about the already-rotated $y$ axis, $R_z(\phi)\boldsymbol{e}_y = \boldsymbol{e}_\phi$. Thus, with all vectors in the same space frame, the reference frame has angular velocity

$$\boldsymbol{\omega}_{\text{ref}} = \dot\phi\,\boldsymbol{e}_z + \dot\theta\,\boldsymbol{e}_\phi.$$

Differentiating the full product for $Q$ adds the residual rotation about the carried axis:

$$\boldsymbol{\omega} = \boldsymbol{\omega}_{\text{ref}} + \dot\chi\,\boldsymbol{n}.$$

Project along $\boldsymbol{n}$. Since $\boldsymbol{e}_z \cdot \boldsymbol{n} = \cos\theta$ and $\boldsymbol{e}_\phi \cdot \boldsymbol{n} = 0$, the physical axial spin is

$$\boxed{\sigma = \boldsymbol{\omega}\cdot\boldsymbol{n} = \dot\chi + \dot\phi\cos\theta.}$$

This formula should be read geometrically rather than memorized as a component identity. The spin along $\boldsymbol{n}$ has two contributions:

- the explicit residual rotation rate $\dot{\chi}$ about the body symmetry axis;
- the rotation about $\boldsymbol{n}$ induced by moving the chosen frame $Q_{\boldsymbol{n}}$ as the axis direction moves on the sphere.

### 13.1 Reconstruct the three-dimensional motion

On a nonsingular coordinate patch, the reconstruction is now explicit:

$$\phi(t) = \phi_0 + \int_{t_0}^{t} \frac{J_z - J_3 \cos\theta(u)}{I_\perp \sin^2\theta(u)}\, du, \tag{4}$$

$$\chi(t) = \chi_0 + \frac{J_3}{I_\parallel}(t - t_0) - \int_{t_0}^{t} \dot{\phi}(u)\cos\theta(u)\, du. \tag{5}$$

First obtain $\theta(t)$ from its initial data and (3), then integrate (4), then (5). Finally construct $Q(t) = R_z(\phi(t))R_y(\theta(t))R_3(\chi(t))$. This maps any body-fixed point or marked axis into the laboratory. In particular $\boldsymbol{r}_{\mathrm{CM}} = \ell\boldsymbol{n}$ follows without solving another dynamical equation.

For the horizontal steady-precession check, $\cos\theta = 0$, so $\phi = \phi_0 + \nu(t - t_0)$ and $\chi = \chi_0 + (J_3/I_\parallel)(t - t_0)$. At a different constant tilt, the same physical spin instead gives $\dot{\chi} = J_3/I_\parallel - \nu\cos\theta_0$. The difference is the axial rotation already supplied by the moving reference frame.

### 13.2 Rechoosing the reference frame does not change the spin

Along a path, let another choice be $Q_{\boldsymbol{n}}^{\mathrm{new}} = Q_{\boldsymbol{n}} R_3(\alpha)$ and $\chi^{\mathrm{new}} = \chi - \alpha$. The angle $\alpha$ may change as the axis moves. The reference frame then gains axial angular velocity $\dot{\alpha}\,\boldsymbol{n}$, while the residual rate loses $\dot{\alpha}$. Hence

$$\boldsymbol{\omega}_{\mathrm{ref}}^{\mathrm{new}} \cdot \boldsymbol{n} + \dot{\chi}^{\mathrm{new}} = (\boldsymbol{\omega}_{\mathrm{ref}} \cdot \boldsymbol{n} + \dot{\alpha}) + (\dot{\chi} - \dot{\alpha}) = \sigma.$$

The product $Q$ and the physical spin are unchanged. This is freedom in the *description* of a single orientation, not a new physical torque or a new independent conserved quantity. It is also why an Euler-angle rate should not be identified with an angular-velocity component without the projection calculation.

The representation hierarchy is therefore

$$\begin{array}{c} Q: \text{ full orientation and direct frame map} \\ \Downarrow \text{ use symmetry} \\ Q = Q_{\boldsymbol{n}} R_3(\chi): \text{ axis direction + residual spin} \\ \Downarrow \text{ choose local coordinates when useful} \\ \boldsymbol{n} \leftrightarrow (\theta, \phi), \qquad \chi \text{ reconstructs the full orientation.} \end{array}$$

## 14 Summary map

**Required rigid-body dynamics.** Use space components for angular-momentum balance and body components for constant inertia:

$$\boxed{\begin{array}{ccc} \dot{\boldsymbol{J}}_s = \boldsymbol{\tau}_s & \Longrightarrow & \boldsymbol{\tau}_b = \dot{\boldsymbol{J}}_b + \boldsymbol{\omega}_b \times \boldsymbol{J}_b, \\ \boldsymbol{J}_b = \mathsf{I}_b \boldsymbol{\omega}_b & \Longrightarrow & \mathsf{I}_b \dot{\boldsymbol{\omega}}_b + \boldsymbol{\omega}_b \times (\mathsf{I}_b \boldsymbol{\omega}_b) = \boldsymbol{\tau}_b. \end{array}}$$

Principal axes simplify the inertia tensor, not the nonlinear dynamics. With zero torque, $E$ and $J^2$ constrain the motion. Linearization gives stable transverse oscillations about the smallest- and largest-moment axes and instability about the intermediate one. For the torque-free symmetric top,

$$\dot{\boldsymbol{n}} = \frac{1}{I_\perp} \boldsymbol{J}_s \times \boldsymbol{n},$$

so its symmetry axis precesses around fixed angular momentum. The body-component rate and the space-axis rate describe different motions.

**Optional extension.** The heavy top uses the same strategy of choosing a useful description:

$$\boxed{\begin{array}{c} Q = Q_{\boldsymbol{n}} R_3(\chi) \quad \Longrightarrow \quad (\boldsymbol{n}, \text{physical axial spin}), \\ J_3,\ J_z,\ E \text{ conserved} \quad \Longrightarrow \quad \dfrac{1}{2} I_\perp \dot{\theta}^2 + V_{\text{eff}}(\theta) = \mathcal{E}, \\ \theta(t) \ \longrightarrow \ \phi(t) \ \longrightarrow \ \chi(t) \ \longrightarrow \ Q(t). \end{array}}$$

The quadrature gives time along an allowed tilt interval. Ordinary turning points, steady precession, and coordinate poles require different interpretations. Reconstruction restores the full orientation, with physical spin $\sigma = \dot{\chi} + \dot{\phi} \cos\theta$ rather than $\dot{\chi}$ alone. All of this heavy-top extension remains outside the Exam II boundary.

After Exam II, Lecture 15 begins the Hamiltonian formulation. The ideas already visible here—conserved momenta, changing the description of a state, and reconstructing motion after reduction—will return in a more systematic language. No Hamiltonian formalism is needed for the calculations in this chapter.