Graduate Classical Mechanics
Lecture 13: Rigid Bodies I

Point matrices, moving frames, angular velocity, and inertia

1 The organizing picture

A rigid body contains many material points, but its internal distances do not change. The central problem of rigid-body kinematics is therefore to separate two kinds of motion:

- motion of the body as a whole through space;
- change of the body's orientation without change of its shape.

The main structural chain of this lecture is

$$\boxed{\begin{array}{c} X = [\boldsymbol{x}_1 \ \cdots \ \boldsymbol{x}_N] \quad \xrightarrow{\text{rigidity}} \quad X(t) = \boldsymbol{R}(t)\mathbf{1}^T + Q(t)X_0 \\ \Downarrow \\ Q^TQ = I, \qquad \dot{Q} = W_sQ, \qquad W_s^T = -W_s \\ \Downarrow \\ W_s\boldsymbol{r} = \boldsymbol{\omega}_s \times \boldsymbol{r}, \qquad \dot{\boldsymbol{x}}_i = \dot{\boldsymbol{R}} + \boldsymbol{\omega}_s \times (\boldsymbol{x}_i - \boldsymbol{R}) \\ \Downarrow \\ T = \tfrac{1}{2}m_T|\dot{\boldsymbol{R}}|^2 + \tfrac{1}{2}\boldsymbol{\omega}_b^T \mathsf{I}_b \boldsymbol{\omega}_b, \qquad \boldsymbol{J}_b = \mathsf{I}_b\boldsymbol{\omega}_b. \end{array}}$$

The point of the lecture is not to begin by assuming a familiar formula such as $\dot{\boldsymbol{x}} = \boldsymbol{\omega} \times \boldsymbol{x}$. Instead, we will let the rigid-body structure force the appropriate representation step by step.

2 A rigid body as a point matrix

Suppose the body is represented by material points with positions

$$\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N \in \mathbb{R}^d.$$

Arrange these vectors as the columns of a matrix:

$$\boxed{X = [\boldsymbol{x}_1 \ \boldsymbol{x}_2 \ \cdots \ \boldsymbol{x}_N] \in \mathbb{R}^{d\times N}.}$$

This is the same collection of point coordinates that we have used before, but organized differently. Earlier, especially in discrete systems, it was often useful to form one long stacked vector

$$\boldsymbol{x}_{\text{stack}} = \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \vdots \\ \boldsymbol{x}_N \end{bmatrix} \in \mathbb{R}^{dN}.$$

That representation is natural when operators act on the list of particles or edges. The present $d \times N$ point matrix is natural when the *same map in physical space* acts on every point from the left.

3 Gram matrices and distance data

Define the Gram matrix of the point configuration:

$$\boxed{\mathsf{G}(X) = X^T X.}$$

Its entries are

$$\boxed{G_{ij} = \boldsymbol{x}_i^T \boldsymbol{x}_j.}$$

The diagonal entries contain squared distances from the chosen origin,

$$G_{ii} = |\boldsymbol{x}_i|^2,$$

while off-diagonal entries contain mutual inner products.

The Gram matrix is useful, but it does not by itself describe the shape of a free body independently of the spatial origin. Translating every point changes the numbers $\boldsymbol{x}_i^T \boldsymbol{x}_j$.

The internal geometry of a rigid body is instead encoded by pairwise distances. Define

$$d_{ij} = |\boldsymbol{x}_j - \boldsymbol{x}_i|.$$

Then

$$\begin{aligned} d_{ij}^2 &= (\boldsymbol{x}_j - \boldsymbol{x}_i)^T (\boldsymbol{x}_j - \boldsymbol{x}_i) \\ &= \boldsymbol{x}_i^T \boldsymbol{x}_i + \boldsymbol{x}_j^T \boldsymbol{x}_j - 2\boldsymbol{x}_i^T \boldsymbol{x}_j. \end{aligned}$$

Let

$$\boxed{\boldsymbol{g} = \mathrm{diag}(X^T X)}$$

denote the column vector of diagonal entries of the Gram matrix. The complete matrix of squared pairwise distances is then

$$\boxed{\mathsf{D}^2(X) = \boldsymbol{g}\mathbf{1}^T + \mathbf{1}\boldsymbol{g}^T - 2X^T X.}$$

This formula makes an important distinction visible:

$$\boxed{\begin{array}{c} X^T X: \text{ depends on the chosen origin,} \\ \mathsf{D}^2(X): \text{ contains internal distances only.} \end{array}}$$

For a free rigid body, the second object is the one that captures shape.

4 Rigid affine maps

Consider a general affine transformation acting on the whole point matrix:

$$\boxed{Y = A\mathbf{1}^T + BX,}$$

where

$$A \in \mathbb{R}^d, \qquad B \in GL(d).$$

Column by column,

$$\boxed{\boldsymbol{y}_i = A + B\boldsymbol{x}_i.}$$

Take the difference between two transformed points:

$$\boxed{\boldsymbol{y}_j - \boldsymbol{y}_i = B(\boldsymbol{x}_j - \boldsymbol{x}_i).}$$

The translation A disappears. Thus translation cannot change any internal distance.

The squared distance after the transformation is

$$\begin{aligned} d_{ij}^2(Y) &= (\boldsymbol{y}_j - \boldsymbol{y}_i)^T(\boldsymbol{y}_j - \boldsymbol{y}_i) \\ &= (\boldsymbol{x}_j - \boldsymbol{x}_i)^T B^T B(\boldsymbol{x}_j - \boldsymbol{x}_i). \end{aligned}$$

Therefore

$$\boxed{B^T B = I_d \quad \Longrightarrow \quad \mathsf{D}^2(Y) = \mathsf{D}^2(X).}$$

If the relative displacement vectors of the body span $\mathbb{R}^d$, the converse also holds. Preserving the lengths of a spanning set of relative vectors and the lengths of their differences preserves their mutual inner products by polarization. Therefore the linear part must preserve the Euclidean inner product on a basis:

$$\boxed{B^T B = I_d.}$$

Thus rigidity forces the linear part of the affine map to be orthogonal. Since

$$B^{-1} = B^T, \qquad \det B = \pm 1,$$

the orientation-preserving rigid motions continuously connected to the identity use

$$\boxed{Q^T Q = QQ^T = I_d, \qquad Q^{-1} = Q^T, \qquad \det Q = +1.}$$

In three dimensions we write $Q \in SO(3)$, but the name is less important than the geometric content: Q preserves lengths, angles, and handedness.

4.1 A geometric qualification

If all the points happen to lie in a lower-dimensional affine subspace, then the distance data constrain the transformation only on the span of the relative displacement vectors. For an ordinary three-dimensional rigid body, we assume enough noncoplanar material points to determine the full orientation.

5 Choose one reference body

Select one configuration as a reference configuration:

$$\boxed{X_0 = X(0).}$$

Choose the reference spatial axes so that initially

$$\boxed{\boldsymbol{R}(0) = \mathbf{0}, \qquad Q(0) = I_d.}$$

Then every rigid configuration can be written as

$$\boxed{X(t) = \boldsymbol{R}(t)\mathbf{1}^T + Q(t)X_0.}$$

Column by column,

$$\boxed{\boldsymbol{x}_i(t) = \boldsymbol{R}(t) + Q(t)\boldsymbol{x}_{i0}.}$$

This is the basic configuration-level description of a rigid body. The reference coordinates $\boldsymbol{x}_{i0}$ are fixed. All time dependence is carried by

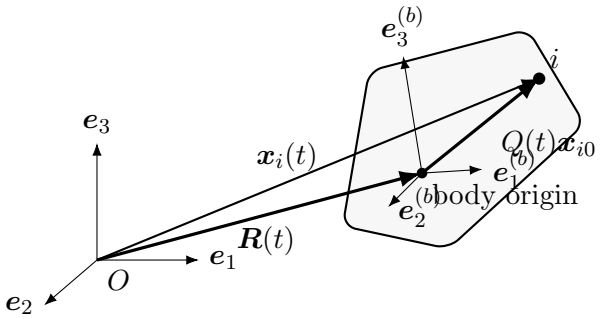
$$\boxed{\boldsymbol{R}(t) \quad \text{and} \quad Q(t).}$$

The vector $\boldsymbol{R}$ locates the chosen body origin in the laboratory frame; Q describes the orientation of the body frame relative to the laboratory frame.

In three dimensions, write the body-attached basis vectors in laboratory components as the columns of Q:

$$\boxed{Q(t) = [\boldsymbol{e}_1^{(b)}(t)\ \boldsymbol{e}_2^{(b)}(t)\ \boldsymbol{e}_3^{(b)}(t)].}$$

Each column is a moving unit vector in physical space.



The same relation can be compressed into the block product

$$\boxed{X(t) = [\boldsymbol{R}(t)\ Q(t)] \begin{bmatrix} \mathbf{1}^T \\ X_0 \end{bmatrix}.}$$

A useful direct check is obtained by subtracting two columns:

$$\boldsymbol{x}_j(t) - \boldsymbol{x}_i(t) = Q(t)(\boldsymbol{x}_{j0} - \boldsymbol{x}_{i0}).$$

Because $Q^TQ = I$, the length of every relative displacement is constant. Thus the configuration formula reproduces the original rigidity requirement.

6 Choosing the body origin: the center of mass

The origin used in the body coordinates is not fixed by geometry alone. We can choose it. The dynamically useful choice is the center of mass.

Introduce the diagonal particle-mass matrix

$$\boxed{\mathsf{M} = \operatorname{diag}(m_1, \ldots, m_N)}$$

and total mass

$$\boxed{m_T = \mathbf{1}^T \mathsf{M} \mathbf{1}.}$$

For a reference configuration with an arbitrary body origin, define the mass-weighted mean of its columns:

$$\boxed{\boldsymbol{c} = \frac{X_0 \mathsf{M} \mathbf{1}}{m_T} \in \mathbb{R}^d.}$$

Recenter every reference point by the same vector:

$$\boxed{X_{0c} = X_0 - \boldsymbol{c}\mathbf{1}^T.}$$

Then

$$\begin{aligned} X_{0c}\mathsf{M}\mathbf{1} &= X_0\mathsf{M}\mathbf{1} - \boldsymbol{c}\,\mathbf{1}^T\mathsf{M}\mathbf{1} \\ &= X_0\mathsf{M}\mathbf{1} - m_T\boldsymbol{c} \\ &= \mathbf{0}. \end{aligned}$$

Therefore, after choosing the center of mass as the body origin and dropping the temporary subscript c, the reference coordinates satisfy

$$\boxed{X_0 \mathsf{M} \mathbf{1} = \mathbf{0}.}$$

This recentering does not change the physical configuration. A change of body origin can be absorbed into a corresponding change of $\boldsymbol{R}$:

$$\boldsymbol{R}\mathbf{1}^T + Q(X_0 + \boldsymbol{c}\mathbf{1}^T) = (\boldsymbol{R} + Q\boldsymbol{c})\mathbf{1}^T + QX_0.$$

Now multiply the rigid-body configuration by $\mathsf{M}\mathbf{1}$:

$$\begin{aligned} X(t)\mathsf{M}\mathbf{1} &= \boldsymbol{R}\mathbf{1}^T\mathsf{M}\mathbf{1} + QX_0\mathsf{M}\mathbf{1} \\ &= m_T\boldsymbol{R}. \end{aligned}$$

Thus

$$\boxed{\boldsymbol{R}(t) = \frac{X(t)\mathsf{M}\mathbf{1}}{\mathbf{1}^T\mathsf{M}\mathbf{1}}.}$$

So with this choice of body origin, $\boldsymbol{R}$ is exactly the center-of-mass position.

The same condition will have a second payoff: it removes the translation–rotation cross term from the kinetic energy.

7 How does the moving frame change?

We now turn from configuration to velocity. The columns of

$$Q = [\boldsymbol{e}_1^{(b)}\ \boldsymbol{e}_2^{(b)}\ \boldsymbol{e}_3^{(b)}]$$

form a basis at every time. Therefore there is a unique linear map that takes each current body axis into its instantaneous time derivative. Call this map W_s:

$$\boxed{\dot{\boldsymbol{e}}_a^{(b)} = W_s \boldsymbol{e}_a^{(b)}, \qquad a = 1, 2, 3.}$$

Collecting the three columns gives

$$\boxed{\dot{Q} = W_s Q.}$$

Since $Q^{-1} = Q^T$,

$$\boxed{W_s = \dot{Q} Q^T.}$$

The subscript s emphasizes that this matrix is written in space, or laboratory, components.

7.1 Rigidity forces the instantaneous generator to be skew

Differentiate the orthogonality condition

$$QQ^T = I_3.$$

Then

$$\dot{Q}Q^T + Q\dot{Q}^T = 0.$$

But

$$\dot{Q}Q^T = W_s$$

and

$$Q\dot{Q}^T = (\dot{Q}Q^T)^T = W_s^T.$$

Therefore

$$\boxed{W_s^T = -W_s.}$$

This is the velocity-level signature of an orthonormal moving frame. The finite orientation Q lies on the nonlinear constraint $Q^TQ = I$; its instantaneous generator W_s lies in the linear space of skew-symmetric matrices.

8 Angular velocity in three dimensions

Every real 3×3 skew-symmetric matrix can be written as

$$W_s = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}.$$

The corresponding vector

$$\boxed{\boldsymbol{\omega}_s = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}}$$

is defined so that

$$\boxed{W_s \boldsymbol{r} = \boldsymbol{\omega}_s \times \boldsymbol{r}.}$$

This gives a structural meaning to angular velocity: $\boldsymbol{\omega}_s$ is the three-component vector representation of the instantaneous skew map that moves the orthonormal body frame.

A finite rotation should not be confused with an angular-velocity vector. Finite rotations do not add as ordinary vectors. Their instantaneous generators, however, form a three-dimensional linear space, and in three dimensions that space can be represented by vectors through the cross product.

For example, if

$$\boldsymbol{\omega}_s = \omega \boldsymbol{e}_3,$$

then

$$W_s \boldsymbol{e}_1 = \omega \boldsymbol{e}_2, \qquad W_s \boldsymbol{e}_2 = -\omega \boldsymbol{e}_1.$$

The basis vectors instantaneously move around the $\boldsymbol{e}_3$ axis in the expected sense.

9 The rigid-body velocity field

Define the displacement of particle i from the center of mass:

$$\boxed{\boldsymbol{r}_i(t) = \boldsymbol{x}_i(t) - \boldsymbol{R}(t) = Q(t)\boldsymbol{x}_{i0}.}$$

Differentiate:

$$\dot{\boldsymbol{r}}_i = \dot{Q}\boldsymbol{x}_{i0}.$$

Using $\dot{Q} = W_s Q$,

$$\dot{\boldsymbol{r}}_i = W_s Q \boldsymbol{x}_{i0} = W_s \boldsymbol{r}_i.$$

Therefore

$$\boxed{\dot{\boldsymbol{r}}_i = \boldsymbol{\omega}_s \times \boldsymbol{r}_i.}$$

Since

$$\boldsymbol{x}_i = \boldsymbol{R} + \boldsymbol{r}_i,$$

we obtain

$$\boxed{\dot{\boldsymbol{x}}_i = \dot{\boldsymbol{R}} + \boldsymbol{\omega}_s \times (\boldsymbol{x}_i - \boldsymbol{R}).}$$

One translational velocity $\dot{\boldsymbol{R}}$ and one angular velocity $\boldsymbol{\omega}_s$ determine the instantaneous velocity of every point of the rigid body.

For the whole point matrix,

$$\boxed{\dot{X} = \dot{\boldsymbol{R}}\mathbf{1}^T + W_s\big(X - \boldsymbol{R}\mathbf{1}^T\big).}$$

A quick check comes from the skew symmetry of W_s:

$$\boldsymbol{r}_i^T \dot{\boldsymbol{r}}_i = \boldsymbol{r}_i^T W_s \boldsymbol{r}_i = 0.$$

Thus each point's velocity relative to the center of mass is perpendicular to its instantaneous radius vector. Equivalently,

$$\frac{d}{dt}|\boldsymbol{r}_i|^2 = 0.$$

This is exactly what rigidity requires.

10 Space components and body components

A single physical vector can be represented in two coordinate systems:

- laboratory, or space, components;
- components along the moving body axes.

If $\boldsymbol{b}_b$ is the body-component column vector and $\boldsymbol{b}_s$ is the laboratory-component column vector of the same physical vector, then

$$\boxed{\boldsymbol{b}_s = Q\boldsymbol{b}_b, \qquad \boldsymbol{b}_b = Q^T\boldsymbol{b}_s.}$$

The angular-velocity generator can likewise be written in body components. Define

$$\boxed{W_b = Q^T\dot{Q}.}$$

Differentiating $Q^TQ = I$ gives

$$\boxed{W_b^T = -W_b.}$$

The two generators satisfy

$$\boxed{\dot{Q} = W_sQ = QW_b,}$$

and hence

$$\boxed{W_b = Q^TW_sQ.}$$

In three dimensions this becomes

$$\boxed{\boldsymbol{\omega}_b = Q^T\boldsymbol{\omega}_s.}$$

The vectors $\boldsymbol{\omega}_s$ and $\boldsymbol{\omega}_b$ do not represent two different angular velocities. They are the laboratory and body coordinate columns of the same physical angular velocity.

The symbol on a vector is not enough to determine its coordinate representation. Always ask: are these components written in the inertial space frame or in the moving body frame?

11 The derivative of a vector in a moving frame

Suppose a physical vector is represented by

$$\boxed{\boldsymbol{b}_s = Q\boldsymbol{b}_b.}$$

Differentiate using the product rule:

$$\dot{\boldsymbol{b}}_s = \dot{Q}\boldsymbol{b}_b + Q\dot{\boldsymbol{b}}_b.$$

Using $\dot{Q} = QW_b$,

$$\dot{\boldsymbol{b}}_s = Q(W_b\boldsymbol{b}_b + \dot{\boldsymbol{b}}_b).$$

Multiply by Q^T:

$$\boxed{Q^T\dot{\boldsymbol{b}}_s = \dot{\boldsymbol{b}}_b + W_b\boldsymbol{b}_b.}$$

In three dimensions,

$$\boxed{Q^T\dot{\boldsymbol{b}}_s = \dot{\boldsymbol{b}}_b + \boldsymbol{\omega}_b \times \boldsymbol{b}_b.}$$

This relation is simply the product rule for a vector expressed in a moving basis. It is not a new dynamical law.

Two limiting cases help fix the meaning.

Vector painted on the body

If the vector is fixed in the body,

$$\dot{\boldsymbol{b}}_b = 0.$$

Then

$$Q^T\dot{\boldsymbol{b}}_s = \boldsymbol{\omega}_b \times \boldsymbol{b}_b.$$

Its laboratory components change because the basis carrying it rotates.

Vector fixed in the laboratory

If the physical vector is constant in the space frame,

$$\dot{\boldsymbol{b}}_s = 0.$$

Then

$$\boxed{\dot{\boldsymbol{b}}_b = -\boldsymbol{\omega}_b \times \boldsymbol{b}_b.}$$

Its body components change because the body axes rotate underneath it.

This moving-frame derivative will be the bridge from the simple inertial equation for angular momentum in space to Euler's equations in Lecture 14.

12 Configuration variables versus velocity data

The rigid configuration is specified by

$$\boxed{(\boldsymbol{R}, Q).}$$

A convenient collection of velocity-level data is

$$\boxed{(\boldsymbol{R}, Q, \dot{\boldsymbol{R}}, \boldsymbol{\omega}).}$$

The distinction matters. Q describes finite orientation; $\boldsymbol{\omega}$ describes its instantaneous rate of change. Angular velocity is not itself an orientation coordinate.

13 Kinetic energy from the particle description

We now let the inertia tensor emerge from the original particle kinetic energy rather than introducing it by definition.

The kinetic energy of the particles can be written compactly in point-matrix form:

$$\boxed{T = \frac{1}{2}\,\mathrm{tr}(\dot{X}\mathsf{M}\dot{X}^T).}$$

Indeed,

$$\mathrm{tr}(\dot{X}\mathsf{M}\dot{X}^T) = \sum_i m_i \dot{\boldsymbol{x}}_i^T \dot{\boldsymbol{x}}_i.$$

From

$$X = \boldsymbol{R}\mathbf{1}^T + QX_0$$

we have

$$\dot{X} = \dot{\boldsymbol{R}}\mathbf{1}^T + \dot{Q}X_0.$$

Substitute into the kinetic energy:

$$T = \frac{1}{2}\,\mathrm{tr}\left[(\dot{\boldsymbol{R}}\mathbf{1}^T + \dot{Q}X_0)\mathsf{M}(\mathbf{1}\dot{\boldsymbol{R}}^T + X_0^T\dot{Q}^T)\right].$$

Expanding the three distinct contributions gives

$$\boxed{T = \frac{1}{2}m_T\dot{\boldsymbol{R}}^T\dot{\boldsymbol{R}} + \dot{\boldsymbol{R}}^T\dot{Q}X_0\mathsf{M}\mathbf{1} + \frac{1}{2}\,\mathrm{tr}(\dot{Q}X_0\mathsf{M}X_0^T\dot{Q}^T).}$$

The mixed term vanishes because the body origin is the center of mass:

$$X_0\mathsf{M}\mathbf{1} = \mathbf{0}.$$

Therefore

$$\boxed{T = \frac{1}{2}m_T|\dot{\boldsymbol{R}}|^2 + T_{\mathrm{rot}},}$$

where

$$\boxed{T_{\text{rot}} = \frac{1}{2}\,\text{tr}(\dot{Q}X_0\mathsf{M}X_0^T\dot{Q}^T).}$$

This is the dynamical reason the center of mass is the natural body origin: the translational and rotational kinetic energies separate exactly. An external potential need not separate in the same way.

14 The second-moment matrix

Define the body-frame second-moment matrix

$$\boxed{\mathsf{S}_b = X_0\mathsf{M}X_0^T.}$$

Since both X_0 and M are fixed, S_b is constant in body coordinates.

Using

$$\dot{Q} = QW_b,$$

we have

$$\begin{aligned} T_{\text{rot}} &= \frac{1}{2}\,\text{tr}(QW_b\mathsf{S}_bW_b^TQ^T) \\ &= \frac{1}{2}\,\text{tr}(W_b\mathsf{S}_bW_b^T), \end{aligned}$$

where we used cyclic invariance of the trace and $Q^TQ = I$. Thus

$$\boxed{T_{\text{rot}} = \frac{1}{2}\,\text{tr}(W_b\mathsf{S}_bW_b^T).}$$

The matrix S_b is the mass-weighted second moment of the reference point cloud. The familiar inertia tensor will now emerge from this object.

15 Deriving the inertia tensor

Return to the particle sum. Since

$$W_b\boldsymbol{x}_{i0} = \boldsymbol{\omega}_b \times \boldsymbol{x}_{i0},$$

the rotational kinetic energy is

$$\boxed{T_{\text{rot}} = \frac{1}{2}\sum_i m_i|\boldsymbol{\omega}_b \times \boldsymbol{x}_{i0}|^2.}$$

Use the vector identity

$$|\boldsymbol{\omega} \times \boldsymbol{x}|^2 = |\boldsymbol{\omega}|^2|\boldsymbol{x}|^2 - (\boldsymbol{\omega} \cdot \boldsymbol{x})^2.$$

In matrix form,

$$\boxed{|\boldsymbol{\omega} \times \boldsymbol{x}|^2 = \boldsymbol{\omega}^T\left(|\boldsymbol{x}|^2 I_3 - \boldsymbol{x}\boldsymbol{x}^T\right)\boldsymbol{\omega}.}$$

Therefore

$$T_{\text{rot}} = \frac{1}{2}\boldsymbol{\omega}_b^T \left[\sum_i m_i \left(|\boldsymbol{x}_{i0}|^2 I_3 - \boldsymbol{x}_{i0}\boldsymbol{x}_{i0}^T\right)\right] \boldsymbol{\omega}_b.$$

Define the body-frame inertia tensor by

$$\boxed{\mathsf{I}_b = \sum_i m_i \left(|\boldsymbol{x}_{i0}|^2 I_3 - \boldsymbol{x}_{i0}\boldsymbol{x}_{i0}^T\right).}$$

Then

$$\boxed{T_{\text{rot}} = \frac{1}{2}\boldsymbol{\omega}_b^T \mathsf{I}_b \boldsymbol{\omega}_b.}$$

Because

$$\mathsf{S}_b = X_0 \mathsf{M} X_0^T = \sum_i m_i \boldsymbol{x}_{i0}\boldsymbol{x}_{i0}^T$$

and

$$\operatorname{tr}(\mathsf{S}_b) = \sum_i m_i |\boldsymbol{x}_{i0}|^2,$$

we obtain the compact relation

$$\boxed{\mathsf{I}_b = \operatorname{tr}(\mathsf{S}_b) I_3 - \mathsf{S}_b.}$$

This formula gives a useful interpretation. The second-moment matrix records how mass is distributed along spatial directions. Rotational inertia about a given axis depends instead on mass distributed *perpendicular* to that axis.

For a unit vector $\hat{\boldsymbol{n}}$,

$$\begin{aligned} I_{\hat{\boldsymbol{n}}} &= \hat{\boldsymbol{n}}^T \mathsf{I}_b \hat{\boldsymbol{n}} \\ &= \sum_i m_i \left(|\boldsymbol{x}_{i0}|^2 - (\hat{\boldsymbol{n}} \cdot \boldsymbol{x}_{i0})^2\right). \end{aligned}$$

Thus

$$\boxed{I_{\hat{\boldsymbol{n}}} = \sum_i m_i r_{i,\perp}^2.}$$

Distance perpendicular to the axis determines rotational inertia.

16 Angular momentum about the center of mass

The angular momentum about the center of mass is

$$\boxed{\boldsymbol{J}_s = \sum_i \boldsymbol{r}_i \times m_i \dot{\boldsymbol{r}}_i.}$$

Using

$$\dot{\boldsymbol{r}}_i = \boldsymbol{\omega}_s \times \boldsymbol{r}_i,$$

we have

$$\boldsymbol{J}_s = \sum_i m_i \boldsymbol{r}_i \times (\boldsymbol{\omega}_s \times \boldsymbol{r}_i).$$

The triple-product identity gives

$$\boldsymbol{r}_i \times (\boldsymbol{\omega}_s \times \boldsymbol{r}_i) = |\boldsymbol{r}_i|^2 \boldsymbol{\omega}_s - \boldsymbol{r}_i(\boldsymbol{r}_i \cdot \boldsymbol{\omega}_s).$$

Therefore

$$\boxed{\boldsymbol{J}_s = \mathsf{I}_s \boldsymbol{\omega}_s,}$$

where I_s is the inertia tensor expressed in the space frame.

The rotational kinetic energy may equivalently be written as

$$\boxed{T_{\text{rot}} = \frac{1}{2} \boldsymbol{\omega}_s \cdot \boldsymbol{J}_s.}$$

Because the mass distribution is fixed in body coordinates,

$$\boxed{\mathsf{I}_s(t) = Q \mathsf{I}_b Q^T.}$$

Transforming angular momentum to body components gives

$$\begin{aligned} \boldsymbol{J}_b &= Q^T \boldsymbol{J}_s \\ &= Q^T \mathsf{I}_s \boldsymbol{\omega}_s \\ &= Q^T (Q \mathsf{I}_b Q^T) \boldsymbol{\omega}_s \\ &= \mathsf{I}_b \boldsymbol{\omega}_b. \end{aligned}$$

Hence

$$\boxed{\boldsymbol{J}_b = \mathsf{I}_b \boldsymbol{\omega}_b.}$$

This is one of the major reasons body coordinates are so useful: the inertia tensor is constant there.

17 What is simple in each frame?

The two descriptions have complementary advantages:

space frame: inertial differentiation is simple,
body frame: X_0, S_b, I_b are constant.

The moving-frame derivative is what lets us move cleanly between them.

This tradeoff is the central setup for rigid-body dynamics. In the space frame, Newton's law for angular momentum has the familiar form

$$\dot{\boldsymbol{J}}_s = \boldsymbol{\tau}_s.$$

But the constitutive relation between angular momentum and angular velocity is simplest in body coordinates:

$$\boldsymbol{J}_b = \mathsf{I}_b \boldsymbol{\omega}_b.$$

Lecture 14 will combine these statements.

18 Principal-axis preview

The inertia tensor is real and symmetric:

$$\boxed{\mathsf{I}_b^T = \mathsf{I}_b.}$$

Therefore the same symmetric-matrix reasoning used in the small-oscillation problem applies again. There exists an orthonormal body basis in which

$$\boxed{\mathsf{I}_b = \mathrm{diag}(I_1, I_2, I_3).}$$

These basis directions are the principal axes and I_1, I_2, I_3 are the principal moments of inertia.

In general,

$$\boxed{\boldsymbol{J} \nparallel \boldsymbol{\omega}.}$$

They are parallel only when $\boldsymbol{\omega}$ lies along a principal axis, or when the relevant principal moments are degenerate.

This fact is the source of much of the interesting dynamics of a freely rotating asymmetric rigid body.

19 What has not been introduced yet

The primary orientation variable in this lecture is the matrix Q. We have not defined orientation by Euler angles. Euler angles can be useful later when a local three-coordinate parameterization is needed, especially for top problems, but they are not required to understand rigid-body kinematics itself.

Likewise, the geometric-algebra description of instantaneous rotation remains available as a parallel language: the skew generator can be represented by an angular-velocity bivector. A full rotor construction is not needed for the present matrix-based derivation.

The important conceptual separation is

$$\boxed{\text{finite orientation } Q \quad \neq \quad \text{instantaneous angular velocity } \boldsymbol{\omega}.}$$

20 Summary map

The rigid-body construction can now be summarized as

$$
\begin{array}{c}
\text{material points} \quad \longrightarrow \quad X = [\boldsymbol{x}_1 \ \cdots \ \boldsymbol{x}_N] \\
\Downarrow \text{ preserve all internal distances} \\
Q^T Q = I, \qquad X(t) = \boldsymbol{R}(t)\mathbf{1}^T + Q(t)X_0 \\
\Downarrow \text{ choose COM body origin} \\
X_0 \mathsf{M}\mathbf{1} = 0, \qquad \boldsymbol{R} = \frac{X\mathsf{M}\mathbf{1}}{\mathbf{1}^T\mathsf{M}\mathbf{1}} \\
\Downarrow \text{ differentiate the moving frame} \\
\dot{Q} = W_s Q = Q W_b, \qquad W_s^T = -W_s \\
\Downarrow \text{ three dimensions} \\
W_s \boldsymbol{r} = \boldsymbol{\omega}_s \times \boldsymbol{r}, \qquad \dot{\boldsymbol{x}}_i = \dot{\boldsymbol{R}} + \boldsymbol{\omega}_s \times (\boldsymbol{x}_i - \boldsymbol{R}) \\
\Downarrow \text{ particle kinetic energy} \\
\mathsf{S}_b = X_0 \mathsf{M} X_0^T, \qquad \mathsf{I}_b = \operatorname{tr}(\mathsf{S}_b) I_3 - \mathsf{S}_b \\
\Downarrow \\
T = \tfrac{1}{2} m_T |\dot{\boldsymbol{R}}|^2 + \tfrac{1}{2} \boldsymbol{\omega}_b^T \mathsf{I}_b \boldsymbol{\omega}_b, \qquad \boldsymbol{J}_b = \mathsf{I}_b \boldsymbol{\omega}_b.
\end{array}
$$

The main structural lesson is that a rigid body becomes simple only after choosing a representation adapted to the constraint. The point matrix makes the common spatial action visible, the rotation matrix carries the finite orientation, the skew generator carries the instantaneous rotation, and the body frame makes the mass distribution constant.

21 The next question

Suppose there is no external torque about the center of mass. In the laboratory frame,

$$\boxed{\dot{\boldsymbol{J}}_s = 0.}$$

Does that imply

$$\dot{\boldsymbol{\omega}}_s = 0?$$

Not in general, because the inertia tensor expressed in the space frame changes as the body rotates, and $\boldsymbol{J}$ need not be parallel to $\boldsymbol{\omega}$.

The next step is to combine

$$\boxed{\dot{\boldsymbol{J}}_s = \boldsymbol{\tau}_s,}$$

with the moving-frame derivative

$$\boxed{Q^T \dot{\boldsymbol{J}}_s = \dot{\boldsymbol{J}}_b + \boldsymbol{\omega}_b \times \boldsymbol{J}_b,}$$

and the body-frame constitutive relation

$$\boxed{\boldsymbol{J}_b = \mathsf{I}_b \boldsymbol{\omega}_b.}$$

That combination leads to Euler's equations and the dynamics of torque-free rigid-body rotation.