Graduate Classical Mechanics
Lecture 12: Small Oscillations II and Beyond Linearization

Mass-weighted modes, matrix evolution, zero modes, and nonlinear phase space

1 The organizing picture

Lecture 11 ended with the general linearized small-oscillation problem

$$\boxed{M\ddot{\boldsymbol{\xi}} + K\boldsymbol{\xi} = \mathbf{0},}$$

and the generalized normal-mode eigenvalue problem

$$\boxed{K\boldsymbol{u}_\alpha = \omega_\alpha^2 M\boldsymbol{u}_\alpha.}$$

Here $\boldsymbol{\xi} = \boldsymbol{q} - \boldsymbol{q}^{(0)}$ is the displacement from a stationary equilibrium, $M = M^T > 0$ is the mass matrix, and $K = K^T$ is the quadratic stiffness matrix, both evaluated at that equilibrium. As in Lecture 11, we are considering a conservative natural system with time-independent kinetic and potential energies. In this quadratic model the matrices are constant. The coordinate transformations below are consequently constant too: differentiating $\boldsymbol{\xi} = U\boldsymbol{a}$ does not produce a $\dot{U}$ term.

The generalized mode equation tells us which displacement patterns can oscillate with a single frequency. It does not yet tell us how an arbitrary initial displacement and velocity are assembled from those patterns. That is the practical problem we will solve. We retain Lecture 11's two-mass spring system and carry it all the way from its matrices to its motion.

The purpose of this lecture is to turn that eigenvalue problem into a complete coordinate transformation. We will carry both the component and matrix descriptions in parallel:

$$\boxed{\begin{array}{ccc} D\mathbf{v}_\alpha = \omega_\alpha^2 \mathbf{v}_\alpha & \Longleftrightarrow & DV = V\Omega^2 \\ \Downarrow & & \Downarrow \\ \boldsymbol{u}_\alpha = M^{-1/2}\mathbf{v}_\alpha & \Longleftrightarrow & U = M^{-1/2}V \\ \Downarrow & & \Downarrow \\ \boldsymbol{\xi} = \displaystyle\sum_\alpha \boldsymbol{u}_\alpha a_\alpha & \Longleftrightarrow & \boldsymbol{\xi} = U\boldsymbol{a} \\ \Downarrow & & \Downarrow \\ \ddot{a}_\alpha + \omega_\alpha^2 a_\alpha = 0 & \Longleftrightarrow & \ddot{\boldsymbol{a}} + \Omega^2\boldsymbol{a} = \mathbf{0}. \end{array}}$$

Once the quadratic dynamics have been separated, we will then ask what the approximation has omitted: mode coupling, amplitude dependence, separatrices, and stability questions that cannot be decided from the quadratic form alone.

2 Mass weighting converts the generalized eigenproblem into a symmetric one

Why rescale the displacements before finding eigenvectors? The kinetic energy assigns a size to a velocity pattern:

$$T^{(2)} = \frac{1}{2}\dot{\boldsymbol{\xi}}^{\,T} M \dot{\boldsymbol{\xi}}.$$

For two Cartesian coordinates with masses m and $2m$, for example, this is $\frac{1}{2}m\dot{\xi}_1^2 + m\dot{\xi}_2^2$. Equal coordinate speeds do not make equal contributions to the kinetic energy. Ordinary Euclidean lengths in the original coordinate column therefore do not express the relevant inertial weighting.

The generalized eigenvalue problem

$$K\boldsymbol{u}_\alpha = \omega_\alpha^2 M \boldsymbol{u}_\alpha$$

contains two symmetric matrices rather than one. Because M is positive definite, it has a unique positive-definite square root $M^{1/2}$. Define the mass-weighted displacement

$$\boxed{\boldsymbol{y} = M^{1/2}\boldsymbol{\xi}.}$$

The square root is a matrix square root, not an entry-by-entry operation. If $M = R\,\mathrm{diag}(\mu_i)R^T$, where $R^T R = I$ and every $\mu_i > 0$, then $M^{1/2} = R\,\mathrm{diag}(\sqrt{\mu_i})R^T$. For diagonal particle masses this is simply the familiar rescaling $y_i = \sqrt{m_i}\,\xi_i$.

The reason for this choice is visible directly in the energies:

$$T^{(2)} = \frac{1}{2}\dot{\boldsymbol{y}}^{\,T} M^{-1/2} M M^{-1/2} \dot{\boldsymbol{y}} = \frac{1}{2}\dot{\boldsymbol{y}}^{\,T}\dot{\boldsymbol{y}},$$
$$V^{(2)} = \frac{1}{2}\boldsymbol{y}^T \big(M^{-1/2} K M^{-1/2}\big)\boldsymbol{y}.$$

The kinetic-energy quadratic form is now Euclidean. We can diagonalize the remaining potential form by an ordinary orthogonal rotation without disturbing that kinetic form. Mass weighting does not remove the physical masses; it puts their effects into the coordinate transformation and the transformed restoring matrix.

We can also see the same change directly in the equations. Since M is constant,

$$\boldsymbol{\xi} = M^{-1/2}\boldsymbol{y}, \qquad \ddot{\boldsymbol{\xi}} = M^{-1/2}\ddot{\boldsymbol{y}}.$$

Substituting into the equations of motion gives

$$M M^{-1/2}\ddot{\boldsymbol{y}} + K M^{-1/2}\boldsymbol{y} = \boldsymbol{0}.$$

Multiplying by $M^{-1/2}$ from the left,

$$\boxed{\ddot{\boldsymbol{y}} + D\boldsymbol{y} = \boldsymbol{0}, \qquad D \equiv M^{-1/2} K M^{-1/2}.}$$

Since M and K are symmetric,

$$\boxed{D = D^T.}$$

Thus the generalized small-oscillation problem has been converted into an ordinary symmetric eigenvalue problem. Multiplication on both sides of K is essential: $M^{-1}K$, although it produces the same frequencies, is generally not symmetric in the original Euclidean coordinates.

Choose a complete orthonormal eigenbasis of D:

$$\boxed{D\mathbf{v}_\alpha = \omega_\alpha^2 \mathbf{v}_\alpha.}$$

Collect the eigenvectors as columns of the matrix

$$\boxed{V = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_N \end{pmatrix}, \qquad V^T V = I.}$$

Define

$$\boxed{\Omega^2 = \mathrm{diag}(\omega_1^2, \ldots, \omega_N^2).}$$

Then all of the component eigenvalue equations are summarized by

$$\boxed{DV = V\Omega^2.}$$

The order matters: the $\mathbf{v}_\alpha$ are the *columns* of V, so the eigenvalues multiply from the right.

For the nonnegative part of the spectrum we choose $\omega_\alpha \geq 0$ and write

$$\boxed{\Omega = \mathrm{diag}(\omega_1, \ldots, \omega_N).}$$

Negative values of ω_α^2 will be interpreted later as linear instability.

3 Return to the original coordinates: the modal matrix U

The mass-weighted eigenvectors $\mathbf{v}_\alpha$ live in the Euclidean $\boldsymbol{y}$-coordinates. Their corresponding mode shapes in the original generalized coordinates are

$$\boxed{\boldsymbol{u}_\alpha = M^{-1/2}\mathbf{v}_\alpha.}$$

Collecting them as columns gives

$$\boxed{U = \begin{pmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 & \cdots & \boldsymbol{u}_N \end{pmatrix} = M^{-1/2}V.}$$

Starting from $DV = V\Omega^2$,

$$M^{-1/2}KM^{-1/2}V = V\Omega^2.$$

Multiplying by $M^{1/2}$,

$$K(M^{-1/2}V) = M(M^{-1/2}V)\Omega^2.$$

Therefore

$$\boxed{KU = MU\Omega^2,}$$

which is the matrix form of

$$\boxed{K\boldsymbol{u}_\alpha = \omega_\alpha^2 M\boldsymbol{u}_\alpha.}$$

This construction also makes the natural orthogonality immediate:

$$\begin{aligned} U^T MU &= V^T M^{-1/2} M M^{-1/2} V \\ &= V^T V \\ &= I. \end{aligned}$$

Hence

$$\boxed{U^T M U = I} \qquad \Longleftrightarrow \qquad \boxed{\boldsymbol{u}_\alpha^T M \boldsymbol{u}_\beta = \delta_{\alpha\beta}.}$$

The normal modes are therefore orthonormal in the M-weighted inner product

$$\boxed{\langle \boldsymbol{u}_\alpha, \boldsymbol{u}_\beta \rangle_M \equiv \boldsymbol{u}_\alpha^T M \boldsymbol{u}_\beta.}$$

Similarly, using $KU = MU\Omega^2$,

$$U^T K U = U^T M U \Omega^2 = \Omega^2,$$

so

$$\boxed{U^T K U = \Omega^2} \qquad \Longleftrightarrow \qquad \boxed{\boldsymbol{u}_\alpha^T K \boldsymbol{u}_\beta = \omega_\alpha^2 \delta_{\alpha\beta}.}$$

Thus the same modal basis simultaneously reduces the kinetic and potential quadratic forms. A mode shape and its normalization are different choices: multiplying an eigenvector by a nonzero constant leaves its frequency and displacement ratios unchanged. The choice $\boldsymbol{u}_\alpha^T M \boldsymbol{u}_\alpha = 1$ puts unit coefficients in the modal kinetic energy. For Cartesian displacements, the resulting $\boldsymbol{u}_\alpha$ have units of inverse square root of mass, and a_α has units of length times square root of mass. A normal coordinate need not itself be a particle displacement.

3.1 Direct component proof of mass orthogonality

The matrix construction is efficient, but the component proof is also useful. Take two modes:

$$K\boldsymbol{u}_\alpha = \omega_\alpha^2 M \boldsymbol{u}_\alpha, \qquad K\boldsymbol{u}_\beta = \omega_\beta^2 M \boldsymbol{u}_\beta.$$

Left-multiply the first by $\boldsymbol{u}_\beta^T$ and the second by $\boldsymbol{u}_\alpha^T$. Since M and K are symmetric,

$$\boxed{(\omega_\alpha^2 - \omega_\beta^2) \boldsymbol{u}_\beta^T M \boldsymbol{u}_\alpha = 0.}$$

For distinct eigenvalues,

$$\boxed{\omega_\alpha^2 \neq \omega_\beta^2 \quad \Longrightarrow \quad \boldsymbol{u}_\beta^T M \boldsymbol{u}_\alpha = 0.}$$

If the eigenvalues are degenerate, the equation imposes no orthogonality condition by itself. Because D is symmetric, however, we may choose an orthonormal basis inside the degenerate eigenspace, which induces an M-orthonormal basis in the original coordinates.

4 Normal coordinates diagonalize the dynamics

Expand the displacement in the mode basis:

$$\boxed{\boldsymbol{\xi} = U\boldsymbol{a}} \qquad \Longleftrightarrow \qquad \boxed{\boldsymbol{\xi} = \sum_{\alpha=1}^{N} \boldsymbol{u}_\alpha a_\alpha.}$$

Here $\boldsymbol{a} = (a_1, \ldots, a_N)^T$ is the vector of normal coordinates.

Substitute $\boldsymbol{\xi} = U\boldsymbol{a}$ into the linearized equations:

$$MU\ddot{\boldsymbol{a}} + KU\boldsymbol{a} = \mathbf{0}.$$

Left-multiply by U^T:

$$U^T MU\ddot{\boldsymbol{a}} + U^T KU\boldsymbol{a} = \mathbf{0}.$$

Using

$$U^T MU = I, \qquad U^T KU = \Omega^2,$$

we obtain

$$\boxed{\ddot{\boldsymbol{a}} + \Omega^2\boldsymbol{a} = \mathbf{0}.}$$

Componentwise this is simply

$$\boxed{\ddot{a}_\alpha + \omega_\alpha^2 a_\alpha = 0.}$$

Thus the original coupled system has become a set of independent scalar equations.

The quadratic kinetic energy becomes

$$\begin{aligned} T^{(2)} &= \frac{1}{2}\dot{\boldsymbol{\xi}}^T M\dot{\boldsymbol{\xi}} \\ &= \frac{1}{2}\dot{\boldsymbol{a}}^T U^T MU\dot{\boldsymbol{a}} \\ &= \boxed{\frac{1}{2}\dot{\boldsymbol{a}}^T\dot{\boldsymbol{a}}} = \boxed{\sum_{\alpha=1}^{N}\frac{1}{2}\dot{a}_\alpha^2}. \end{aligned}$$

The quadratic potential energy becomes

$$\begin{aligned} V^{(2)} &= \frac{1}{2}\boldsymbol{\xi}^T K\boldsymbol{\xi} \\ &= \frac{1}{2}\boldsymbol{a}^T U^T KU\boldsymbol{a} \\ &= \boxed{\frac{1}{2}\boldsymbol{a}^T\Omega^2\boldsymbol{a}} = \boxed{\sum_{\alpha=1}^{N}\frac{1}{2}\omega_\alpha^2 a_\alpha^2}. \end{aligned}$$

Hence

$$\boxed{L^{(2)} = \frac{1}{2}\dot{\boldsymbol{a}}^T\dot{\boldsymbol{a}} - \frac{1}{2}\boldsymbol{a}^T\Omega^2\boldsymbol{a} = \sum_{\alpha=1}^{N}\left[\frac{1}{2}\dot{a}_\alpha^2 - \frac{1}{2}\omega_\alpha^2 a_\alpha^2\right].}$$

The generalized eigenproblem therefore does more than find frequencies: it constructs coordinates in which both quadratic forms become diagonal.

5 Projecting the initial state into modal coordinates

Since $\boldsymbol{\xi} = U\boldsymbol{a}$ and $U^T MU = I$, multiplication by $U^T M$ recovers the modal amplitudes directly:

$$\boxed{\boldsymbol{a} = U^T M\boldsymbol{\xi}.}$$

Componentwise,

$$\boxed{a_\alpha = \boldsymbol{u}_\alpha^T M \boldsymbol{\xi}.}$$

Therefore the complete initial state projects as

$$\boxed{\boldsymbol{a}(0) = U^T M \boldsymbol{\xi}(0), \qquad \dot{\boldsymbol{a}}(0) = U^T M \dot{\boldsymbol{\xi}}(0),}$$

or, for each mode,

$$\boxed{a_\alpha(0) = \boldsymbol{u}_\alpha^T M \boldsymbol{\xi}(0), \qquad \dot{a}_\alpha(0) = \boldsymbol{u}_\alpha^T M \dot{\boldsymbol{\xi}}(0).}$$

A mode decomposition acts on the full dynamical state: both the initial displacement and the initial velocity are required. An initially undisplaced mode can still be excited by its initial velocity.

It is worth checking why the inverse is $U^T M$, not U^T. For the complete square modal matrix, $U^T M U = I$ says precisely that

$$\boxed{U^{-1} = U^T M.}$$

Equivalently, first mass-weight the configuration, then project it onto the Euclidean orthonormal columns of V:

$$\boldsymbol{a} = V^T \boldsymbol{y} = V^T M^{1/2} \boldsymbol{\xi} = U^T M \boldsymbol{\xi}.$$

Using $U^T \boldsymbol{\xi}$ alone would project with the wrong inner product. The inverse formula assumes that all N independent modes are retained; a truncated set gives a projection, not an exact reconstruction of an arbitrary state.

6 Positive-frequency evolution: component and matrix forms

For a mode with $\omega_\alpha > 0$, the scalar initial-value problem

$$\ddot{a}_\alpha + \omega_\alpha^2 a_\alpha = 0$$

has the solution

$$\boxed{a_\alpha(t) = a_\alpha(0)\cos(\omega_\alpha t) + \frac{\dot{a}_\alpha(0)}{\omega_\alpha}\sin(\omega_\alpha t).}$$

If every mode in the block has strictly positive frequency, the same result is written compactly as

$$\boxed{\boldsymbol{a}(t) = \cos(\Omega t)\boldsymbol{a}(0) + \Omega^{-1}\sin(\Omega t)\dot{\boldsymbol{a}}(0).}$$

Since Ω is diagonal, the matrix functions are simply

$$\cos(\Omega t) = \operatorname{diag}\big(\cos\omega_1 t, \ldots, \cos\omega_N t\big),$$

$$\Omega^{-1}\sin(\Omega t) = \operatorname{diag}\left(\frac{\sin\omega_1 t}{\omega_1}, \ldots, \frac{\sin\omega_N t}{\omega_N}\right).$$

The initial conditions are built into this form. At $t = 0$, the cosine matrix is I and the sine matrix is zero, so the first term returns the initial displacement and the second vanishes. Differentiating,

$$\dot{\boldsymbol{a}}(t) = -\Omega\sin(\Omega t)\boldsymbol{a}(0) + \cos(\Omega t)\dot{\boldsymbol{a}}(0),$$

which gives the correct $\dot{\boldsymbol{a}}(0)$. Differentiating again yields

$$\ddot{\boldsymbol{a}}(t) = -\Omega^2 \boldsymbol{a}(t).$$

7 Worked example: unequal masses from modes to motion

Return to Lecture 11's two moving masses between fixed walls, joined by three identical springs. To make every stage explicit, choose

$$\text{wall} - \kappa - m - \kappa - 2m - \kappa - \text{wall}, \qquad m > 0, \quad \kappa > 0.$$

The displacements ξ_1, ξ_2 are measured from equilibrium. Their energies are

$$T = \frac{1}{2} m \dot{\xi}_1^2 + \frac{1}{2}(2m)\dot{\xi}_2^2, \qquad V = \frac{\kappa}{2}\left[\xi_1^2 + (\xi_2 - \xi_1)^2 + \xi_2^2\right].$$

Consequently

$$M = m \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \qquad K = \kappa \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

For this ideal one-dimensional Hooke-spring model, these energies are exactly quadratic. The modal solution is exact within that model; for a general system they would instead be the local quadratic approximation.

7.1 Find frequencies and displacement ratios

Mass weighting gives

$$M^{1/2} = \sqrt{m} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{pmatrix}, \qquad D = \frac{\kappa}{m} \begin{pmatrix} 2 & -1/\sqrt{2} \\ -1/\sqrt{2} & 1 \end{pmatrix}.$$

Define the dimensionless squared frequency $\lambda = m\omega^2/\kappa$. The characteristic equation is

$$\det \begin{pmatrix} 2-\lambda & -1/\sqrt{2} \\ -1/\sqrt{2} & 1-\lambda \end{pmatrix} = (2-\lambda)(1-\lambda) - \frac{1}{2} = \lambda^2 - 3\lambda + \frac{3}{2} = 0.$$

Thus the lower- and higher-frequency modes, labeled $-$ and $+$, have

$$\boxed{\omega_\pm^2 = \frac{\kappa}{m}\,\frac{3 \pm \sqrt{3}}{2}.}$$

Both are positive. The fixed walls remove the uniform-translation zero mode of a free chain.

To find the physical displacement ratios, use the first row of $K\boldsymbol{u} = \omega^2 M\boldsymbol{u}$:

$$(2-\lambda)u_1 - u_2 = 0.$$

Choose unnormalized shapes

$$\mathbf{e}_\pm = \begin{pmatrix} 1 \\ r_\pm \end{pmatrix}, \qquad r_- = \frac{1+\sqrt{3}}{2}, \qquad r_+ = \frac{1-\sqrt{3}}{2}.$$

In the lower mode, both masses move in the same direction, with the heavier mass having the larger displacement. In the higher mode they move in opposite directions. Unlike the equal-mass case, the patterns are not $(1,1)^T$ and $(1,-1)^T$.

7.2 Normalize in the correct geometry

Since $r_- r_+ = -1/2$,

$$\mathbf{e}_-^T \mathbf{e}_+ = \frac{1}{2} \neq 0, \qquad \mathbf{e}_-^T M \mathbf{e}_+ = m(1 + 2r_- r_+) = 0.$$

The modes are not Euclidean-orthogonal, but they are mass-orthogonal. Their mass-normalization factors are

$$\mathcal{N}_- = \sqrt{m(3+\sqrt{3})}, \qquad \mathcal{N}_+ = \sqrt{m(3-\sqrt{3})}, \qquad \mathcal{N}_\pm^2 = m(1+2r_\pm^2).$$

Hence

$$\boxed{\boldsymbol{u}_\pm = \frac{1}{\mathcal{N}_\pm}\begin{pmatrix}1\\ r_\pm\end{pmatrix}, \qquad U = \begin{pmatrix}\boldsymbol{u}_- & \boldsymbol{u}_+\end{pmatrix}.}$$

The corresponding mass-weighted vectors are

$$\mathbf{v}_\pm = M^{1/2}\boldsymbol{u}_\pm = \frac{1}{\sqrt{1+2r_\pm^2}}\begin{pmatrix}1\\ \sqrt{2}r_\pm\end{pmatrix}.$$

They satisfy ordinary Euclidean orthonormality, while U satisfies

$$U^T M U = I, \qquad U^T K U = \mathrm{diag}(\omega_-^2, \omega_+^2).$$

We now have not just two frequencies but an invertible coordinate transformation.

7.3 Project a complete initial state

Choose an initial displacement of the lighter mass and an initial velocity of the heavier mass:

$$\boxed{\boldsymbol{\xi}(0) = \begin{pmatrix}d\\ 0\end{pmatrix}, \qquad \dot{\boldsymbol{\xi}}(0) = \begin{pmatrix}0\\ v\end{pmatrix}.}$$

Here d is a prescribed displacement and v a prescribed velocity component, with either sign allowed. These data are not a single normal-mode motion. Projection gives

$$\boxed{a_\pm(0) = \frac{md}{\mathcal{N}_\pm}, \qquad \dot{a}_\pm(0) = \frac{2mr_\pm v}{\mathcal{N}_\pm}.}$$

The factor of two in the second expression is the heavier particle's mass. It would be lost by an unweighted projection. Each initial coordinate pattern generally excites both modes.

7.4 Evolve and reconstruct

The independent mode solutions are

$$a_\pm(t) = \frac{md}{\mathcal{N}_\pm}\cos(\omega_\pm t) + \frac{2mr_\pm v}{\mathcal{N}_\pm \omega_\pm}\sin(\omega_\pm t).$$

Return to the particle displacements using the same mode shapes:

$$\xi_1(t) = \frac{a_-(t)}{\mathcal{N}_-} + \frac{a_+(t)}{\mathcal{N}_+}, \qquad \xi_2(t) = \frac{r_- a_-(t)}{\mathcal{N}_-} + \frac{r_+ a_+(t)}{\mathcal{N}_+}.$$

Substitution produces an explicit answer in the physical coordinates:

$$\boxed{\begin{aligned}\xi_1(t) = \frac{d}{6}\left[(3-\sqrt{3})\cos(\omega_- t) + (3+\sqrt{3})\cos(\omega_+ t)\right] \\ + \frac{v}{\sqrt{3}}\left[\frac{\sin(\omega_- t)}{\omega_-} - \frac{\sin(\omega_+ t)}{\omega_+}\right],\end{aligned}}$$

$$\boxed{\begin{aligned}\xi_2(t) = \frac{d}{2\sqrt{3}}\left[\cos(\omega_- t) - \cos(\omega_+ t)\right] \\ + \frac{v}{6}\left[(3+\sqrt{3})\frac{\sin(\omega_- t)}{\omega_-} + (3-\sqrt{3})\frac{\sin(\omega_+ t)}{\omega_+}\right].\end{aligned}}$$

The two particle coordinates are not independent oscillators: each contains both frequencies. Independence belongs to a_- and a_+, not to the original displacement components.

7.5 Check the answer in the original mechanics

At $t = 0$, the cosine terms give $(\xi_1, \xi_2) = (d, 0)$. Their derivatives vanish, while the sine terms give $(\dot{\xi}_1, \dot{\xi}_2) = (0, v)$. Also, the original force equations predict

$$\ddot{\xi}_1(0) = -\frac{2\kappa d}{m}, \qquad \ddot{\xi}_2(0) = \frac{\kappa d}{2m},$$

which follow by twice differentiating the reconstructed solution. More generally, because each column obeys $K\boldsymbol{u}_\alpha = \omega_\alpha^2 M\boldsymbol{u}_\alpha$, the whole reconstructed trajectory obeys $M\ddot{\boldsymbol{\xi}} + K\boldsymbol{\xi} = 0$.

Energy provides a check on the normalization. In the physical coordinates the initial energy is

$$E = \kappa d^2 + mv^2.$$

In the normal coordinates it is the sum of the constant modal energies,

$$E = E_- + E_+, \qquad E_\pm = \frac{1}{2}\left[\dot{a}_\pm(0)^2 + \omega_\pm^2 a_\pm(0)^2\right].$$

These expressions agree upon inserting the projected data. The cross terms disappear because the modes were normalized with M, not because the physical spring coupling has ceased to exist.

This calculation exhibits the entire procedure: determine the mode shapes, normalize them, project *both* pieces of the initial state, evolve scalar amplitudes, and reconstruct the original configuration. An eigenvalue list alone accomplishes only the first part of that job.

8 Zero modes and the null-space-resolved propagator

A zero-frequency mode must be treated differently. If

$$\omega_0 = 0,$$

then

$$\ddot{a}_0 = 0,$$

so

$$\boxed{a_0(t) = a_0(0) + \dot{a}_0(0)t.}$$

A zero-frequency mode is therefore not necessarily static; it may describe uniform drift along a flat or symmetry direction. The positive-frequency formula has the same continuous limit, because

$$\lim_{\omega\to 0} \cos(\omega t) = 1, \qquad \lim_{\omega\to 0} \frac{\sin(\omega t)}{\omega} = t.$$

There is no physical divergence here. The difficulty lies only in writing Ω^{-1} for a matrix with a zero eigenvalue. Replacing that inverse by a pseudoinverse alone would incorrectly erase the drift term; the null-space contribution must still be retained.

This statement describes the quadratic model. For an exact translational symmetry, uniform drift can also be an exact motion. For a generic marginal equilibrium, a drift predicted by linearization can eventually leave the small neighborhood where the approximation is valid. We will return to that distinction in Section 13.

For a nonnegative spectrum, order the columns of U so that the positive-frequency modes appear first and the zero modes last:

$$\boxed{U = \begin{pmatrix} U_+ & U_0 \end{pmatrix}, \qquad \Omega = \begin{pmatrix} \Omega_+ & 0 \\ 0 & 0 \end{pmatrix}, \qquad \boldsymbol{a} = \begin{pmatrix} \boldsymbol{a}_+ \\ \boldsymbol{a}_0 \end{pmatrix}, \qquad \Omega_+ > 0.}$$

Here U_0 spans the zero-mode subspace. The complete modal evolution can then be written without any singular inverse:

$$\boxed{\boldsymbol{a}(t) = \begin{pmatrix} \cos(\Omega_+ t) & 0 \\ 0 & I_0 \end{pmatrix} \boldsymbol{a}(0) + \begin{pmatrix} \Omega_+^{-1}\sin(\Omega_+ t) & 0 \\ 0 & tI_0 \end{pmatrix} \dot{\boldsymbol{a}}(0).}$$

The upper block is the usual oscillatory propagator. The lower block is free motion in the null space.

Together with

$$\boxed{\boldsymbol{a}(0) = U^T M \boldsymbol{\xi}(0), \qquad \dot{\boldsymbol{a}}(0) = U^T M \dot{\boldsymbol{\xi}}(0), \qquad \boldsymbol{\xi}(t) = U\boldsymbol{a}(t),}$$

this gives the complete solution in the original generalized coordinates.

The same result in explicit indexed form is

$$\boxed{\begin{aligned} \boldsymbol{\xi}(t) = &\sum_{\omega_\alpha > 0} \boldsymbol{u}_\alpha \left[a_\alpha(0)\cos(\omega_\alpha t) + \frac{\dot{a}_\alpha(0)}{\omega_\alpha}\sin(\omega_\alpha t) \right] \\ &+ \sum_{\omega_\alpha = 0} \boldsymbol{u}_\alpha \left[a_\alpha(0) + \dot{a}_\alpha(0) t \right]. \end{aligned}}$$

Optional computational viewpoint

The zero-mode block can also be written using a Moore–Penrose pseudoinverse together with the projector onto the null space, or by using the analytic matrix function

$$t\,\operatorname{sinc}(\Omega t), \qquad \operatorname{sinc} x = \frac{\sin x}{x}, \qquad \operatorname{sinc} 0 = 1.$$

Those forms are useful computationally and conceptually, but the explicit block form already contains the essential structure without requiring additional machinery.

9 Degeneracy and basis freedom

Degeneracy and zero frequency are different ideas. A degenerate eigenspace contains several independent directions with the same eigenvalue. A zero mode has eigenvalue $\omega^2 = 0$. A zero eigenspace may itself be degenerate, but degeneracy does not imply zero frequency.

The simplest example is an isotropic two-dimensional oscillator:

$$\boxed{M = mI, \qquad K = kI.}$$

Then every nonzero vector $\boldsymbol{u}$ satisfies

$$K\boldsymbol{u} = \frac{k}{m} M\boldsymbol{u},$$

so every direction has

$$\boxed{\omega^2 = \frac{k}{m}.}$$

There is no unique preferred pair of modal directions. Any rotated orthonormal basis spans the same eigenspace.

In matrix language, suppose U_d contains an M-orthonormal basis for a degenerate block with frequency ω. For any orthogonal matrix R,

$$\boxed{U_d \longrightarrow U_d R, \qquad R^T R = I.}$$

Then

$$\boxed{(U_d R)^T M (U_d R) = I,}$$

and

$$\boxed{(U_d R)^T K (U_d R) = \omega^2 I.}$$

To describe the same physical displacement after this basis change, the normal coordinates must change in the opposite way:

$$\boldsymbol{a}_d \longrightarrow R^T \boldsymbol{a}_d, \qquad (U_d R)(R^T \boldsymbol{a}_d) = U_d \boldsymbol{a}_d.$$

The same rotation applies to the initial modal velocities. Thus the individual basis vectors and their amplitude components change, but the reconstructed motion, the degenerate subspace, and its reduced dynamics do not. In the isotropic example, a mass-normalized physical basis has columns $\hat{\mathbf{x}}/\sqrt{m}$ and $\hat{\mathbf{y}}/\sqrt{m}$, or any rotation of that pair.

This is an important general lesson: the eigenspace is physical; a particular orthonormal basis inside a degenerate eigenspace is a choice.

10 Continuum version: weighted orthogonality becomes an integral

The finite-dimensional modal inner product is

$$\boxed{\langle \boldsymbol{u}_\alpha, \boldsymbol{u}_\beta \rangle_M = \boldsymbol{u}_\alpha^T M \boldsymbol{u}_\beta.}$$

In a continuum system, the mass matrix is replaced by a mass density or weight function. Assume boundary conditions for which the spatial mode problem is self-adjoint in the weighted inner product and has a complete discrete set of real modes; these are the conditions under which the following sum representation applies. For a one-dimensional field with density $\rho(x)$, normalized modes satisfy

$$\boxed{\int_0^\ell \rho(x) u_\alpha(x) u_\beta(x)\, dx = \delta_{\alpha\beta}.}$$

A discrete modal expansion

$$\boldsymbol{\xi}(t) = \sum_\alpha \boldsymbol{u}_\alpha a_\alpha(t)$$

becomes the continuum spectral expansion

$$\boxed{\phi(x,t) = \sum_\alpha a_\alpha(t) u_\alpha(x).}$$

Multiplying the expansion by $\rho(x)u_\beta(x)$ and integrating eliminates all other modes. The analog of $\boldsymbol{a} = U^T M \boldsymbol{\xi}$ is therefore

$$\boxed{a_\beta(t) = \int_0^\ell \rho(x) u_\beta(x) \phi(x,t)\, dx.}$$

The same integral with $\dot{\phi}(x,0)$ gives $\dot{a}_\beta(0)$. A field, like a finite-dimensional configuration, needs both its initial value and its initial rate of change to determine its motion.

Thus finite-dimensional normal modes and continuum mode expansions are the same structural move: expand the configuration in a basis adapted to the dynamical operator and its kinetic-energy weight.

11 What the quadratic approximation has discarded

The exact mechanics near an equilibrium contain more than the quadratic Lagrangian

$$L^{(2)} = \frac{1}{2}\dot{\boldsymbol{\xi}}^T M \dot{\boldsymbol{\xi}} - \frac{1}{2}\boldsymbol{\xi}^T K \boldsymbol{\xi}.$$

Schematically,

$$\boxed{L = L^{(2)} + L^{(3)} + L^{(4)} + \cdots.}$$

For the potential,

$$\boxed{V = V_0 + \frac{1}{2}K_{ij}\xi_i\xi_j + \frac{1}{3!}V^{(3)}_{ijk}\xi_i\xi_j\xi_k + \frac{1}{4!}V^{(4)}_{ijkl}\xi_i\xi_j\xi_k\xi_l + \cdots.}$$

In general coordinates, the mass matrix $M(\boldsymbol{q})$ also depends on configuration. Expanding it can generate higher-order kinetic terms such as

$$\xi\,\dot{\xi}^{\,2}.$$

These terms were consistently neglected in the quadratic theory.

Transforming to modal coordinates does not make the nonlinear terms disappear. For example, a cubic potential term becomes schematically

$$\boxed{V^{(3)} = \sum_{\alpha\beta\gamma} C_{\alpha\beta\gamma} a_\alpha a_\beta a_\gamma.}$$

Thus

quadratic theory: independent modes and superposition
nonlinear theory: mode coupling and amplitude dependence.

The modal basis remains useful, but the modal amplitudes no longer evolve independently.

11.1 A local example: an initially absent mode is driven

To see what coupling means, consider two mass-normalized coordinates with a local potential expansion

$$V(a_1, a_2) = \frac{1}{2}\omega_1^2 a_1^2 + \frac{1}{2}\omega_2^2 a_2^2 + \gamma a_1^2 a_2, \qquad T = \frac{1}{2}\dot{a}_1^2 + \frac{1}{2}\dot{a}_2^2.$$

Here $\omega_1, \omega_2 > 0$, and γ is a coupling coefficient with the units required to make the last term an energy. This is an illustrative local Taylor model, not an additional term in the ideal Hooke-spring system of Section 7. A cubic truncation need not describe a bounded potential far from the equilibrium, and we use it only in its small-amplitude regime.

Differentiate the potential with respect to each normal coordinate:

$$\boxed{\begin{aligned} \ddot{a}_1 + \omega_1^2 a_1 &= -2\gamma a_1 a_2, \\ \ddot{a}_2 + \omega_2^2 a_2 &= -\gamma a_1^2. \end{aligned}}$$

Suppose $a_1(0) = A$, with $\dot{a}_1(0) = a_2(0) = \dot{a}_2(0) = 0$. The quadratic theory would keep mode 2 identically absent. The nonlinear equations instead give

$$\ddot{a}_2(0) = -\gamma A^2.$$

For nonzero γA, mode 2 starts to move even though it was neither displaced nor given an initial velocity. The modal transformation still diagonalizes the quadratic terms, but it has not diagonalized the full nonlinear mechanics.

The quadratic modal energies are no longer separately conserved. Writing

$$E_\alpha = \frac{1}{2}\dot{a}_\alpha^2 + \frac{1}{2}\omega_\alpha^2 a_\alpha^2,$$

we find

$$\dot{E}_1 = -2\gamma a_1 a_2 \dot{a}_1, \qquad \dot{E}_2 = -\gamma a_1^2 \dot{a}_2.$$

Nevertheless,

$$\frac{d}{dt}\left(E_1 + E_2 + \gamma a_1^2 a_2\right) = 0.$$

Energy is conserved by the full time-independent model, including its interaction term. What is lost is the separation into independent quadratic mode energies. No perturbation expansion or solution of the nonlinear equations is needed to see that distinction.

12 The nonlinear pendulum as a global phase-space example

The exact pendulum equation is

$$\boxed{\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0.}$$

Its conserved energy is

$$\boxed{E = \frac{1}{2}m\ell^2\dot{\theta}^2 + mg\ell(1-\cos\theta).}$$

Near $\theta = 0$,

$$1 - \cos\theta = \frac{1}{2}\theta^2 - \frac{1}{24}\theta^4 + \cdots,$$

so the quadratic approximation gives

$$\boxed{\ddot{\theta} + \frac{g}{\ell}\theta = 0, \qquad \omega_0 = \sqrt{\frac{g}{\ell}}.}$$

This correctly describes sufficiently small oscillations near the stable equilibrium. It does not describe the global phase portrait.

Solving the exact energy relation for $\dot{\theta}$ gives

$$\boxed{\dot{\theta} = \pm\sqrt{\frac{2}{m\ell^2}\left[E - mg\ell(1-\cos\theta)\right]}.}$$

These curves organize the phase plane. The expression inside the square root must be nonnegative. For energy below the upright potential maximum, this confines the angle between turning points where $\dot{\theta} = 0$. Above that maximum the velocity never vanishes, so its sign cannot reverse and the pendulum rotates continuously. Figure 1 draws those energy curves rather than using ellipses outside the small-angle regime.

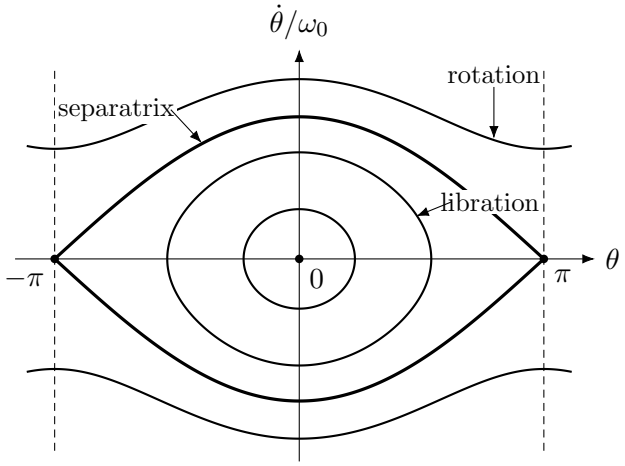

Figure 1: Constant-energy curves of the exact pendulum. With $e = E/(mg\ell)$ and $w = \dot\theta/\omega_0$, they satisfy $w^2/2 + 1 - \cos\theta = e$. The inner curves have $e = 0.245$ and 1.125, the bold separatrix has $e = 2$, and the rotational curves have $e = 3.2$. The lines $\theta = \pm\pi$ bound one angular period, not physical walls.

The energy classification is

$$\boxed{\begin{aligned} 0 < E < 2mg\ell &\implies \text{libration: closed phase curves,} \\ E = 2mg\ell &\implies \text{separatrix through } (\pi, 0), \\ E > 2mg\ell &\implies \text{continuous rotation.} \end{aligned}}$$

A nonconstant separatrix trajectory approaches an upright equilibrium asymptotically; it does not arrive at $(\pi, 0)$ and then leave it in finite time. The upright equilibrium itself is also a solution at this energy. Since angles differing by 2π represent the same configuration, the physical phase space is a cylinder; the displayed plane is an unfolded angular picture.

The harmonic approximation sees only the local center near $(\theta, \dot\theta) = (0, 0)$. It cannot reveal the separatrix or the rotational trajectories because those are global features of the nonlinear system.

The first nonlinear correction to the potential is negative:

$$\boxed{V(\theta) = \frac{1}{2}mg\ell\theta^2 - \frac{1}{24}mg\ell\theta^4 + \cdots .}$$

The pendulum therefore softens as the oscillation amplitude increases; the period is no longer amplitude independent.

13 Marginal stability: when the quadratic theory is inconclusive

Consider

$$\boxed{V_+(x) = \frac{\lambda}{4}x^4, \qquad V_-(x) = -\frac{\lambda}{4}x^4, \qquad \lambda > 0.}$$

For both potentials,

$$\left.\frac{dV}{dx}\right|_{x=0} = 0, \qquad \left.\frac{d^2V}{dx^2}\right|_{x=0} = 0.$$

Thus both have the same quadratic stiffness: zero. Yet their stability is opposite:

$$\boxed{V_+ : \text{ stable minimum}, \qquad V_- : \text{ unstable maximum.}}$$

Higher-order terms decide what the quadratic theory cannot.

For V_+, the conserved energy $m\dot{x}^2/2 + \lambda x^4/4$ bounds both $|x|$ and $|\dot{x}|$ when the energy is small, giving stability of the equilibrium. For V_-, a small positive displacement released from rest has $m\ddot{x} = \lambda x^3 > 0$ and accelerates away. The same zero quadratic curvature therefore conceals opposite nonlinear behavior.

For a normal mode,

$$K\boldsymbol{u} = \omega^2 M\boldsymbol{u}.$$

Multiplying from the left by $\boldsymbol{u}^T$,

$$\boxed{\omega^2 = \frac{\boldsymbol{u}^T K\boldsymbol{u}}{\boldsymbol{u}^T M\boldsymbol{u}}.}$$

Because $M > 0$, the sign of ω^2 is controlled by the quadratic restoring form $\boldsymbol{u}^T K\boldsymbol{u}$. This leads to the stability hierarchy

$$\boxed{K > 0 \quad \Longrightarrow \quad \text{strict quadratic minimum and small oscillations,}}$$

$$\boxed{K \text{ has a negative direction} \quad \Longrightarrow \quad \text{some } \omega^2 < 0 \text{ and linear instability,}}$$

and

$$\boxed{K \geq 0 \text{ with a nontrivial null space} \quad \Longrightarrow \quad \text{higher-order or symmetry information is needed.}}$$

Here $K \geq 0$ means positive semidefinite. A negative eigenvalue already establishes linear instability even if other eigenvalues vanish. Also, a vector with $\boldsymbol{u}^T K\boldsymbol{u} = 0$ need not be a null vector when K is indefinite; the relevant zero mode satisfies $K\boldsymbol{u} = 0$.

A zero eigenvalue can therefore have two very different origins:

- an exact continuous symmetry may produce a genuinely flat direction and a true symmetry zero mode;
- a marginal equilibrium may have zero quadratic curvature even though higher-order terms eventually stabilize or destabilize the motion.

14 Summary map

The complete quadratic construction can be summarized as

$$M\ddot{\boldsymbol{\xi}} + K\boldsymbol{\xi} = 0$$
$$\Downarrow \text{ mass weight}$$
$$D = M^{-1/2}KM^{-1/2}, \qquad DV = V\Omega^2$$
$$\Downarrow$$
$$U = M^{-1/2}V, \qquad U^T MU = I, \qquad U^T KU = \Omega^2$$
$$\Downarrow$$
$$\boldsymbol{\xi} = U\boldsymbol{a}, \qquad \ddot{\boldsymbol{a}} + \Omega^2\boldsymbol{a} = 0$$
$$\Downarrow \text{ restore omitted terms}$$

mode coupling, amplitude dependence, separatrices, and global dynamics.

The practical sequence is

normalize modes $\longrightarrow$ project the initial state $\longrightarrow$ evolve $\longrightarrow$ reconstruct.

The unequal-mass example shows how each step enters an actual solution. None can be replaced by listing the frequencies alone.

The quadratic theory is powerful because it reduces a coupled many-coordinate system to independent modal coordinates. Its limitations are equally important: zero quadratic curvature may require further analysis, and nonlinear terms restore interactions between the modes and introduce global phase-space structures that no local harmonic approximation can see.

The next lecture begins rigid-body mechanics. We will again choose coordinates that simplify kinetic-energy structure, but the central restriction will be exact rigidity rather than a local small-amplitude approximation.