Graduate Classical Mechanics
Lecture 10: Central Forces I

Reduction by coordinates, symmetry, and the effective potential

1 The central-force problem

Consider two particles with masses m_1 and m_2 and position vectors

$$\boxed{\mathbf{x}_1(t), \qquad \mathbf{x}_2(t).}$$

We assume that the particles interact only with each other and that the interaction energy depends on their separation:

$$\boxed{V = V(r), \qquad r = \|\mathbf{x}_2 - \mathbf{x}_1\|.}$$

The original kinetic energy is

$$\boxed{T = \frac{1}{2}m_1\dot{\mathbf{x}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{x}}_2^2.}$$

Thus the Lagrangian in the original particle coordinates is

$$\boxed{L = \frac{1}{2}m_1\dot{\mathbf{x}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{x}}_2^2 - V(|\mathbf{x}_2 - \mathbf{x}_1|).}$$

There are two immediate structural observations.

First, the kinetic energy distinguishes the two particle coordinates individually. Second, the potential does not: it depends only on one particular combination of the coordinates, their difference. This strongly suggests that the original coordinates are not the coordinates best adapted to the interaction.

The goal of this lecture is not simply to solve a special orbit. It is to reorganize the mechanics so that the independent pieces of the motion become visible:

$$\boxed{\begin{array}{c}(\mathbf{x}_1, \mathbf{x}_2) \longrightarrow (\mathbf{R}, \mathbf{r}) \\ \Downarrow \\ \text{overall translation} + \text{internal relative motion} \\ \Downarrow \\ \text{rotational symmetry} \to \text{planar motion} \to \text{one-dimensional radial dynamics.}\end{array}}$$

This reduction is exact. No small-angle, small-amplitude, heavy-mass, or weak-interaction approximation is involved.

2 Let the interaction suggest the first new coordinate

Because the interaction already contains the difference

$$\mathbf{x}_2 - \mathbf{x}_1,$$

define the relative coordinate

$$\boxed{\mathbf{r} \equiv \mathbf{x}_2 - \mathbf{x}_1.}$$

Its magnitude is

$$\boxed{r = \|\mathbf{r}\|,}$$

so the potential immediately becomes

$$\boxed{V = V(r).}$$

This is the first important coordinate choice: choose a coordinate that makes the physical structure of the interaction simple.

However, replacing two vector coordinates by the single vector $\mathbf{r}$ would lose information. We need a second vector coordinate so that the transformation is invertible. The standard and physically natural choice is the center of mass.

Define

$$\boxed{M = m_1 + m_2}$$

and

$$\boxed{\mathbf{R} = \frac{m_1\mathbf{x}_1 + m_2\mathbf{x}_2}{M}.}$$

In the language of generalized coordinates, we began with the particle coordinates

$$(\mathbf{x}_1, \mathbf{x}_2)$$

and have chosen a new pair of generalized coordinates adapted to the mechanics:

$$\boxed{(\mathbf{x}_1, \mathbf{x}_2) \longrightarrow (\mathbf{R}, \mathbf{r}).}$$

Rather than calling these new coordinates $\mathbf{q}_1$ and $\mathbf{q}_2$, we give them names that describe their physical meaning.

2.1 Invert the coordinate transformation

Starting from

$$\mathbf{r} = \mathbf{x}_2 - \mathbf{x}_1, \qquad \mathbf{R} = \frac{m_1\mathbf{x}_1 + m_2\mathbf{x}_2}{M},$$

solve for the original positions.

Since

$$\mathbf{x}_2 = \mathbf{x}_1 + \mathbf{r},$$

we have

$$\mathbf{R} = \frac{m_1\mathbf{x}_1 + m_2(\mathbf{x}_1 + \mathbf{r})}{M} = \mathbf{x}_1 + \frac{m_2}{M}\mathbf{r}.$$

Therefore

$$\boxed{\mathbf{x}_1 = \mathbf{R} - \frac{m_2}{M}\mathbf{r}.}$$

Then

$$\boxed{\mathbf{x}_2 = \mathbf{R} + \frac{m_1}{M}\mathbf{r}.}$$

Thus the transformation is explicitly invertible:

$$\boxed{\begin{aligned} &\mathbf{R} = \frac{m_1\mathbf{x}_1 + m_2\mathbf{x}_2}{M}, \qquad \mathbf{r} = \mathbf{x}_2 - \mathbf{x}_1, \\ &\mathbf{x}_1 = \mathbf{R} - \frac{m_2}{M}\mathbf{r}, \qquad\quad \mathbf{x}_2 = \mathbf{R} + \frac{m_1}{M}\mathbf{r}. \end{aligned}}$$

3 The kinetic energy separates

Differentiate the inverse transformation:

$$\boxed{\dot{\mathbf{x}}_1 = \dot{\mathbf{R}} - \frac{m_2}{M}\dot{\mathbf{r}}, \qquad \dot{\mathbf{x}}_2 = \dot{\mathbf{R}} + \frac{m_1}{M}\dot{\mathbf{r}}.}$$

Now return to the original kinetic energy,

$$T = \frac{1}{2}m_1\dot{\mathbf{x}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{x}}_2^2,$$

and substitute:

$$T = \frac{1}{2}m_1\left(\dot{\mathbf{R}} - \frac{m_2}{M}\dot{\mathbf{r}}\right)^2 + \frac{1}{2}m_2\left(\dot{\mathbf{R}} + \frac{m_1}{M}\dot{\mathbf{r}}\right)^2.$$

Expanding gives

$$\begin{aligned} T = &\frac{1}{2}(m_1 + m_2)\dot{\mathbf{R}}^2 \\ &+ \left(-\frac{m_1m_2}{M} + \frac{m_1m_2}{M}\right)\dot{\mathbf{R}}\cdot\dot{\mathbf{r}} \\ &+ \frac{1}{2}\left[m_1\left(\frac{m_2}{M}\right)^2 + m_2\left(\frac{m_1}{M}\right)^2\right]\dot{\mathbf{r}}^2. \end{aligned}$$

The mixed term cancels exactly.

The remaining coefficient of $\dot{\mathbf{r}}^2$ is

$$\begin{aligned} m_1\frac{m_2^2}{M^2} + m_2\frac{m_1^2}{M^2} &= \frac{m_1m_2(m_1 + m_2)}{M^2} \\ &= \frac{m_1m_2}{M}. \end{aligned}$$

Define the reduced mass

$$\boxed{\mu \equiv \frac{m_1m_2}{M}.}$$

Then

$$\boxed{T = \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2.}$$

The comparison is worth keeping in view:

$$\boxed{\frac{1}{2}m_1\dot{\mathbf{x}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{x}}_2^2} \quad\longrightarrow\quad \boxed{\frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2.}$$

The transformed kinetic energy has no

$$\dot{\mathbf{R}}\cdot\dot{\mathbf{r}}$$

coupling. Overall translation and internal relative motion are kinetically independent in these coordinates.

3.1 How should we interpret the reduced mass?

The reduced mass satisfies

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}.$$

Several limits help interpret it.

If

$$m_2 \gg m_1,$$

then

$$\mu \longrightarrow m_1.$$

The heavier particle moves very little, so the relative motion approaches that of the light particle moving around an effectively fixed center.

If

$$m_1 = m_2 = m,$$

then

$$M = 2m, \qquad \mu = \frac{m}{2}.$$

A change in the separation generally involves motion of both particles, so the inertia associated with the relative coordinate is smaller than the inertia of either particle moving alone while the other is held fixed.

4 Alternative derivation: find the collective coordinate from the kinetic energy

The direct derivation above is the main route. It starts from the physically obvious relative coordinate and then introduces the familiar center of mass.

There is another useful structural question:

Once $\mathbf{r} = \mathbf{x}_2 - \mathbf{x}_1$ has been chosen to simplify the potential, can we determine the second linear coordinate by requiring that the kinetic energy have no mixed relative–collective term?

This does *not* mean that the original kinetic-energy matrix needed to be diagonalized. In the particle coordinates it was already diagonal. Instead, we are asking for a new coordinate system adapted to the potential that also preserves a clean separation of the kinetic energy.

4.1 Determine the second coordinate

Suppose

$$\boxed{\mathbf{R} = a\mathbf{x}_1 + b\mathbf{x}_2.}$$

If both particles are translated by the same constant vector $\mathbf{c}$,

$$\mathbf{x}_1 \to \mathbf{x}_1 + \mathbf{c}, \qquad \mathbf{x}_2 \to \mathbf{x}_2 + \mathbf{c},$$

we want the collective coordinate to transform as an ordinary position:

$$\mathbf{R} \to \mathbf{R} + \mathbf{c}.$$

Therefore

$$\boxed{a + b = 1.}$$

Using

$$\mathbf{r} = \mathbf{x}_2 - \mathbf{x}_1,$$

the inverse transformation for general a, b with $a + b = 1$ is

$$\boxed{\mathbf{x}_1 = \mathbf{R} - b\mathbf{r}, \qquad \mathbf{x}_2 = \mathbf{R} + a\mathbf{r}.}$$

Hence

$$\dot{\mathbf{x}}_1 = \dot{\mathbf{R}} - b\dot{\mathbf{r}}, \qquad \dot{\mathbf{x}}_2 = \dot{\mathbf{R}} + a\dot{\mathbf{r}}.$$

Substitution into

$$T = \frac{1}{2} m_1 \dot{\mathbf{x}}_1^2 + \frac{1}{2} m_2 \dot{\mathbf{x}}_2^2$$

gives

$$\begin{aligned} T = \frac{1}{2}(m_1 + m_2)\dot{\mathbf{R}}^{\,2} + (-m_1 b + m_2 a)\,\dot{\mathbf{R}} \cdot \dot{\mathbf{r}} \\ + \frac{1}{2}(m_1 b^2 + m_2 a^2)\dot{\mathbf{r}}^{\,2}. \end{aligned}$$

Demand that the mixed term vanish:

$$\boxed{-m_1 b + m_2 a = 0.}$$

Together with

$$a + b = 1,$$

this gives

$$\boxed{a = \frac{m_1}{M}, \qquad b = \frac{m_2}{M}.}$$

Therefore

$$\boxed{\mathbf{R} = \frac{m_1 \mathbf{x}_1 + m_2 \mathbf{x}_2}{M}.}$$

The same condition also gives

$$m_1 b^2 + m_2 a^2 = \frac{m_1 m_2}{M} = \mu.$$

Thus the center of mass and the reduced mass can be discovered from the requirement that the relative coordinate and the collective coordinate be kinetically uncoupled.

4.2 Block-matrix form

The same calculation can be compressed into a matrix statement.

Collect the original coordinates into

$$\mathbf{X} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix}$$

and the new coordinates into

$$\mathbf{q} = \begin{pmatrix} \mathbf{R} \\ \mathbf{r} \end{pmatrix}.$$

Let I denote the identity matrix on physical space. The inverse transformation is

$$\boxed{\mathbf{X} = B\mathbf{q},}$$

with

$$\boxed{B = \begin{pmatrix} I & -\dfrac{m_2}{M} I \\ I & \dfrac{m_1}{M} I \end{pmatrix}.}$$

In the original particle coordinates,

$$T = \frac{1}{2} \dot{\mathbf{X}}^T \mathsf{M}_x \dot{\mathbf{X}},$$

where

$$\boxed{\mathsf{M}_x = \begin{pmatrix} m_1 I & 0 \\ 0 & m_2 I \end{pmatrix}.}$$

Again, M_x is already diagonal. The point is not to diagonalize it. The point is to ask what happens to the kinetic quadratic form under the coordinate change required by the interaction.

Since

$$\dot{\mathbf{X}} = B\dot{\mathbf{q}},$$

we have

$$T = \frac{1}{2} \dot{\mathbf{q}}^T B^T \mathsf{M}_x B \dot{\mathbf{q}}.$$

A direct multiplication gives

$$\boxed{B^T \begin{pmatrix} m_1 I & 0 \\ 0 & m_2 I \end{pmatrix} B = \begin{pmatrix} MI & 0 \\ 0 & \mu I \end{pmatrix}.}$$

Therefore

$$\boxed{T = \frac{1}{2} M \dot{\mathbf{R}}^{\,2} + \frac{1}{2} \mu \dot{\mathbf{r}}^{\,2}.}$$

This matrix equation expresses a geometric fact about configuration space: the center-of-mass and relative directions are orthogonal with respect to the mass-weighted kinetic-energy inner product.

The hierarchy of ideas is therefore

$$\boxed{\begin{array}{c} \text{interaction suggests } \mathbf{r} \;\longrightarrow\; \text{choose a complementary } \mathbf{R} \\ \Downarrow \\ \text{require kinetic decoupling} \;\longrightarrow\; \text{center of mass and reduced mass.} \end{array}}$$

5 The Lagrangian splits into overall and internal motion

Because the potential depends only on r,

$$V = V(r),$$

the full Lagrangian becomes

$$\boxed{L = \frac{1}{2}M\dot{\mathbf{R}}^{\,2} + \left[\frac{1}{2}\mu\dot{\mathbf{r}}^{\,2} - V(r)\right].}$$

Define

$$\boxed{L_{\rm cm} = \frac{1}{2}M\dot{\mathbf{R}}^{\,2},}$$

and

$$\boxed{L_{\rm rel} = \frac{1}{2}\mu\dot{\mathbf{r}}^{\,2} - V(r).}$$

Then

$$\boxed{L = L_{\rm cm} + L_{\rm rel}.}$$

The two-body problem has therefore separated into two dynamically independent parts:

center of mass	relative motion
$\mathbf{R}$	$\mathbf{r}$
M	μ
free translation	motion in $V(r)$.

This is a major reduction, but it does not discard any information. Once $\mathbf{R}(t)$ and $\mathbf{r}(t)$ are known, the original particle trajectories are reconstructed exactly from

$$\mathbf{x}_1 = \mathbf{R} - \frac{m_2}{M}\mathbf{r}, \qquad \mathbf{x}_2 = \mathbf{R} + \frac{m_1}{M}\mathbf{r}.$$

6 Translation symmetry and the center of mass

The Lagrangian has no explicit dependence on $\mathbf{R}$:

$$\boxed{\frac{\partial L}{\partial \mathbf{R}} = \mathbf{0}.}$$

Thus $\mathbf{R}$ is a cyclic vector coordinate. Its conjugate momentum is

$$\boxed{\mathbf{P} = \frac{\partial L}{\partial \dot{\mathbf{R}}} = M\dot{\mathbf{R}}.}$$

The Euler–Lagrange equation gives

$$\boxed{\dot{\mathbf{P}} = \mathbf{0}.}$$

Therefore

$$\boxed{\ddot{\mathbf{R}} = \mathbf{0}.}$$

This is total linear-momentum conservation:

$$\begin{aligned} m_1\dot{\mathbf{x}}_1 + m_2\dot{\mathbf{x}}_2 &= m_1\left(\dot{\mathbf{R}} - \frac{m_2}{M}\dot{\mathbf{r}}\right) + m_2\left(\dot{\mathbf{R}} + \frac{m_1}{M}\dot{\mathbf{r}}\right) \\ &= M\dot{\mathbf{R}}. \end{aligned}$$

Hence

$$\boxed{\mathbf{P} = m_1\dot{\mathbf{x}}_1 + m_2\dot{\mathbf{x}}_2 = M\dot{\mathbf{R}}.}$$

This is the Lecture 7 symmetry story in a particularly useful form. Common translation

$$\mathbf{x}_1 \to \mathbf{x}_1 + \mathbf{a}, \qquad \mathbf{x}_2 \to \mathbf{x}_2 + \mathbf{a}$$

leaves the separation $\mathbf{r}$ unchanged. The interaction therefore cannot depend on the absolute position $\mathbf{R}$, and the momentum conjugate to $\mathbf{R}$ is conserved.

We may now choose the center-of-mass frame. In that frame,

$$\boxed{\mathbf{P} = \mathbf{0}}$$

and therefore

$$\boxed{\mathbf{R} = \text{constant}.}$$

The remaining nontrivial mechanics is entirely contained in

$$\boxed{L_{\text{rel}} = \frac{1}{2}\mu\dot{\mathbf{r}}^{\,2} - V(r).}$$

7 The relative equation of motion

Applying the Euler–Lagrange equations to the relative coordinate gives the motion of one effective particle of mass μ in the central potential $V(r)$.

Because

$$r = \|\mathbf{r}\|,$$

a change $d\mathbf{r}$ gives

$$dr = \hat{\mathbf{r}} \cdot d\mathbf{r}, \qquad \hat{\mathbf{r}} \equiv \frac{\mathbf{r}}{r}.$$

Therefore

$$\boxed{\frac{\partial V}{\partial \mathbf{r}} = \frac{dV}{dr}\hat{\mathbf{r}}.}$$

The relative Euler–Lagrange equation is

$$\boxed{\mu\ddot{\mathbf{r}} = -\frac{dV}{dr}\hat{\mathbf{r}}.}$$

The force is central because it is parallel or antiparallel to $\mathbf{r}$.

7.1 How the pair forces appear

The same central-force structure is visible directly in the original particle coordinates. Since

$$\mathbf{r} = \mathbf{x}_2 - \mathbf{x}_1,$$

we have

$$\frac{\partial r}{\partial \mathbf{x}_2} = \hat{\mathbf{r}}, \qquad \frac{\partial r}{\partial \mathbf{x}_1} = -\hat{\mathbf{r}}.$$

Hence

$$\boxed{\mathbf{F}_2 = -\frac{\partial V}{\partial \mathbf{x}_2} = -\frac{dV}{dr}\hat{\mathbf{r}},}$$

while

$$\boxed{\mathbf{F}_1 = -\frac{\partial V}{\partial \mathbf{x}_1} = +\frac{dV}{dr}\hat{\mathbf{r}}.}$$

Thus

$$\boxed{\mathbf{F}_1 = -\mathbf{F}_2,}$$

and both forces lie along the line joining the particles.

A potential of the form $V(|\mathbf{x}_2 - \mathbf{x}_1|)$ therefore supplies both equal-and-opposite internal forces and a central line of action. The first property gives cancellation of internal force in the total momentum balance; the second gives cancellation of internal torque about the center of mass.

8 Rotational symmetry and angular momentum

The relative Lagrangian

$$L_{\text{rel}} = \frac{1}{2}\mu\dot{\mathbf{r}}^{\,2} - V(r)$$

is unchanged by a common spatial rotation of $\mathbf{r}$ and $\dot{\mathbf{r}}$. By the Noether result of Lecture 7, rotational symmetry produces a conserved angular momentum.

Define

$$\boxed{\mathbf{J} = \mathbf{r} \times \mu\dot{\mathbf{r}}.}$$

We use L for the Lagrangian and reserve $\mathbf{J}$ for angular momentum.

8.1 Direct torque check

Differentiate:

$$\begin{aligned}\dot{\mathbf{J}} &= \dot{\mathbf{r}} \times \mu\dot{\mathbf{r}} + \mathbf{r} \times \mu\ddot{\mathbf{r}} \\ &= \mathbf{0} + \mathbf{r} \times \left(-\frac{dV}{dr}\hat{\mathbf{r}}\right).\end{aligned}$$

Since $\mathbf{r}$ is parallel to $\hat{\mathbf{r}}$,

$$\boxed{\dot{\mathbf{J}} = \mathbf{0}.}$$

8.2 Noether check

Take an infinitesimal rotation by a small parameter ϵ about a fixed unit vector $\mathbf{n}$. The symmetry variation of the relative coordinate is

$$\delta_{\text{sym}}\mathbf{r} = \epsilon\, \mathbf{n} \times \mathbf{r}.$$

The conjugate momentum is

$$\mathbf{p}_r = \frac{\partial L_{\text{rel}}}{\partial \dot{\mathbf{r}}} = \mu \dot{\mathbf{r}}.$$

The Noether charge for this one-parameter rotation is

$$\begin{aligned} \mathbf{p}_r \cdot \frac{\delta_{\text{sym}}\mathbf{r}}{\epsilon} &= \mu \dot{\mathbf{r}} \cdot (\mathbf{n} \times \mathbf{r}) \\ &= \mathbf{n} \cdot (\mathbf{r} \times \mu \dot{\mathbf{r}}) \\ &= \boxed{\mathbf{n} \cdot \mathbf{J}}. \end{aligned}$$

Because the axis $\mathbf{n}$ was arbitrary, rotational invariance gives conservation of the vector $\mathbf{J}$.

The torque calculation and the Noether calculation are not competing explanations. They are two formulations of the same structure.

9 Conserved angular momentum makes the orbit planar

By definition,

$$\mathbf{J} \cdot \mathbf{r} = 0$$

and

$$\mathbf{J} \cdot \dot{\mathbf{r}} = 0.$$

If

$$\mathbf{J} \neq \mathbf{0},$$

then $\mathbf{J}$ is a fixed nonzero vector. Therefore the entire relative trajectory lies in the fixed plane through the origin perpendicular to $\mathbf{J}$.

Thus

$$\boxed{\text{3D relative motion} \longrightarrow \text{2D planar motion.}}$$

This is another exact reduction produced by symmetry.

If

$$\mathbf{J} = \mathbf{0},$$

then

$$\mathbf{r} \times \dot{\mathbf{r}} = \mathbf{0},$$

so $\dot{\mathbf{r}}$ is parallel to $\mathbf{r}$. The motion remains on one fixed line through the origin. The planar problem has collapsed to the degenerate purely radial case.

9.1 Total angular momentum

For completeness, the total angular momentum about the original origin can be decomposed into center-of-mass and internal pieces:

$$\boxed{\mathbf{J}_{\text{tot}} = \mathbf{R} \times \mathbf{P} + \mathbf{J}.}$$

Thus $\mathbf{J}$ is the angular momentum of the relative motion about the center of mass. In the center-of-mass frame,

$$\mathbf{P} = \mathbf{0},$$

so

$$\mathbf{J}_{\text{tot}} = \mathbf{J}.$$

10 Polar coordinates in the orbital plane

For nonzero $\mathbf{J}$, choose polar coordinates (r, θ) in the orbital plane:

$$\mathbf{r} = r\,\hat{\mathbf{r}}.$$

The unit vectors satisfy

$$\dot{\hat{\mathbf{r}}} = \dot{\theta}\,\hat{\boldsymbol{\theta}}, \qquad \dot{\hat{\boldsymbol{\theta}}} = -\dot{\theta}\,\hat{\mathbf{r}}.$$

Therefore

$$\begin{aligned} \dot{\mathbf{r}} &= \frac{d}{dt}\,(r\hat{\mathbf{r}}) \\ &= \dot{r}\,\hat{\mathbf{r}} + r\dot{\hat{\mathbf{r}}} \\ &= \boxed{\dot{r}\,\hat{\mathbf{r}} + r\dot{\theta}\,\hat{\boldsymbol{\theta}}.} \end{aligned}$$

Since the polar unit vectors are orthonormal,

$$\boxed{\dot{\mathbf{r}}^{\,2} = \dot{r}^2 + r^2\dot{\theta}^2.}$$

The relative kinetic energy becomes

$$\boxed{T_{\text{rel}} = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\theta}^2,}$$

and the relative Lagrangian is

$$\boxed{L_{\text{rel}} = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r).}$$

The physical meaning of the two terms is transparent:

$$\frac{1}{2}\mu\dot{r}^2$$

is radial kinetic energy, while

$$\frac{1}{2}\mu r^2\dot{\theta}^2$$

is angular kinetic energy.

11 The angle is cyclic

The central potential depends only on r, so

$$\boxed{\frac{\partial L_{\text{rel}}}{\partial \theta} = 0.}$$

Therefore θ is cyclic.

Its conjugate momentum is

$$\boxed{p_\theta = \frac{\partial L_{\text{rel}}}{\partial \dot{\theta}} = \mu r^2 \dot{\theta}.}$$

The Euler–Lagrange equation gives

$$\boxed{\dot{p}_\theta = 0.}$$

Choose the positive normal to the orbital plane so that the signed scalar angular momentum is

$$\boxed{J \equiv p_\theta = \mu r^2 \dot{\theta}.}$$

This is the component of the vector angular momentum $\mathbf{J}$ along the chosen normal.

Thus the chain

$$\boxed{\text{rotational symmetry} \longrightarrow \theta \text{ cyclic} \longrightarrow p_\theta = J \longrightarrow J = \text{constant}}$$

is the concrete central-force realization of the symmetry ideas from Lecture 7.

12 Constant areal velocity

In a small time interval, the relative vector sweeps an infinitesimal triangular area

$$dA = \frac{1}{2} r^2 \, d\theta.$$

Therefore

$$\dot{A} = \frac{1}{2} r^2 \dot{\theta}.$$

Using

$$J = \mu r^2 \dot{\theta},$$

we find

$$\boxed{\dot{A} = \frac{J}{2\mu} = \text{constant}.}$$

Equal areas in equal times therefore follow from angular-momentum conservation for *any* central force. This property is not specific to inverse-square gravity. In the next lecture it will reappear as Kepler's second law for the gravitational potential.

Notice what is and is not constant. Since

$$\dot{\theta} = \frac{J}{\mu r^2},$$

the angular speed is generally not constant. The motion speeds up angularly when r decreases and slows down when r increases. What remains constant is the areal sweep rate.

13 Energy of the relative motion

The relative Lagrangian has no explicit time dependence, so the corresponding Lagrangian energy is conserved. Here the kinetic energy is quadratic in the velocities and the potential is velocity independent, so the conserved Lagrangian energy is the ordinary mechanical energy:

$$\boxed{E_{\text{rel}} = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\theta}^2 + V(r).}$$

Now use angular-momentum conservation:

$$\boxed{\dot{\theta} = \frac{J}{\mu r^2}.}$$

Then

$$\frac{1}{2}\mu r^2\dot{\theta}^2 = \frac{1}{2}\mu r^2\left(\frac{J}{\mu r^2}\right)^2 = \boxed{\frac{J^2}{2\mu r^2}.}$$

Thus

$$\boxed{E_{\text{rel}} = \frac{1}{2}\mu\dot{r}^2 + \frac{J^2}{2\mu r^2} + V(r).}$$

Define the effective potential

$$\boxed{V_{\text{eff}}(r) \equiv V(r) + \frac{J^2}{2\mu r^2}.}$$

The radial energy equation becomes

$$\boxed{E_{\text{rel}} = \frac{1}{2}\mu\dot{r}^2 + V_{\text{eff}}(r).}$$

This has exactly the form of a one-dimensional energy equation.

13.1 What the angular-momentum term means

The term

$$\boxed{\frac{J^2}{2\mu r^2}}$$

is sometimes called the centrifugal potential or angular-momentum barrier. Its origin should be kept clear.

It is *not* a new microscopic interaction between the particles. The original interaction remains $V(r)$. The additional term is the angular kinetic energy

$$\frac{1}{2}\mu r^2\dot{\theta}^2$$

rewritten as a function of r after the conserved value of J has been used to eliminate $\dot{\theta}$.

Thus the effective potential belongs to the *reduced radial description*. It packages the influence of the already-conserved angular motion into a one-dimensional radial problem.

When

$$J \neq 0,$$

$$\boxed{\frac{J^2}{2\mu r^2} \longrightarrow +\infty \qquad (r \to 0^+).}$$

A finite-energy barrier against reaching $r = 0$ follows when the *total* effective potential satisfies

$$\boxed{V(r) + \frac{J^2}{2\mu r^2} \longrightarrow +\infty \qquad (r \to 0^+).}$$

Indeed, the nonnegative radial kinetic energy requires $E_{\text{rel}} \geq V_{\text{eff}}(r)$. The barrier condition holds for the central spring below and the attractive potential $V = -k/r$ with $k > 0$. It is not automatic for every central potential: an attractive singularity can cancel or dominate the positive $1/r^2$ term. For example, $V = -\kappa/r^4$ with $\kappa > 0$ instead makes $V_{\text{eff}} \to -\infty$. Nonzero angular momentum by itself therefore does not establish the barrier for an unrestricted $V(r)$.

When

$$J = 0,$$

the angular-momentum term disappears and the motion is purely radial:

$$V_{\text{eff}}(r) = V(r).$$

14 The radial equation from Euler–Lagrange

The effective-potential energy equation already organizes the radial motion, but it is useful to verify that the Euler–Lagrange equation gives the same one-dimensional dynamics.

Starting from

$$L_{\text{rel}} = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\theta}^2 - V(r),$$

we have

$$\frac{\partial L_{\text{rel}}}{\partial \dot{r}} = \mu\dot{r},$$

so

$$\frac{d}{dt}\left(\frac{\partial L_{\text{rel}}}{\partial \dot{r}}\right) = \mu\ddot{r}.$$

Also,

$$\frac{\partial L_{\text{rel}}}{\partial r} = \mu r\dot{\theta}^2 - \frac{dV}{dr}.$$

Therefore

$$\boxed{\mu\ddot{r} = \mu r\dot{\theta}^2 - \frac{dV}{dr}.}$$

Using

$$\dot{\theta} = \frac{J}{\mu r^2},$$

we obtain

$$\boxed{\mu\ddot{r} = \frac{J^2}{\mu r^3} - \frac{dV}{dr}.}$$

Now differentiate the effective potential:

$$\frac{dV_{\text{eff}}}{dr} = \frac{dV}{dr} - \frac{J^2}{\mu r^3}.$$

Hence

$$\boxed{\mu\ddot{r} = -\frac{dV_{\text{eff}}}{dr}.}$$

Once the conserved value of J has been specified, the radial coordinate obeys a closed one-dimensional equation.

The angle is then reconstructed from

$$\boxed{\dot{\theta} = \frac{J}{\mu r^2}.}$$

Equivalently,

$$\boxed{\theta(t) - \theta(t_0) = \int_{t_0}^{t} \frac{J}{\mu r(t')^2}\, dt'.}$$

The original planar problem has therefore become

$$\boxed{\text{one radial equation} + \text{one reconstruction integral for } \theta.}$$

15 Reading dynamics from the effective potential

The radial energy equation is

$$\frac{1}{2}\mu\dot{r}^2 = E_{\text{rel}} - V_{\text{eff}}(r).$$

Because

$$\dot{r}^2 \geq 0,$$

the dynamically allowed values of r satisfy

$$\boxed{V_{\text{eff}}(r) \leq E_{\text{rel}}.}$$

15.1 Turning points

A radial turning point occurs when

$$\dot{r} = 0.$$

Therefore

$$\boxed{V_{\text{eff}}(r_{\text{turn}}) = E_{\text{rel}}.}$$

The particle can move only between or beyond roots of this equation that border allowed regions.

15.2 Circular relative motion

A circular orbit has constant radius:

$$\dot{r} = 0, \qquad \ddot{r} = 0.$$

The radial equation then gives

$$\boxed{\left.\frac{dV_{\text{eff}}}{dr}\right|_{r=r_*} = 0.}$$

Thus circular motion corresponds to an equilibrium point of the effective one-dimensional problem.

15.3 Radial stability

If the circular radius is a local minimum,

$$\boxed{\left.\frac{d^2V_{\text{eff}}}{dr^2}\right|_{r=r_*} > 0,}$$

then a sufficiently small radial displacement produces a restoring tendency. The circular orbit is radially stable.

If instead

$$\left.\frac{d^2V_{\text{eff}}}{dr^2}\right|_{r=r_*} < 0,$$

the extremum is a local maximum and small radial perturbations grow away from the circular radius.

The effective potential therefore lets us classify important features of the motion before solving the orbit explicitly.

16 Example: a central spring

Consider

$$\boxed{V(r) = \frac{1}{2}K(r - r_0)^2.}$$

Then

$$\boxed{V_{\text{eff}}(r) = \frac{1}{2}K(r - r_0)^2 + \frac{J^2}{2\mu r^2}.}$$

For nonzero J,

$$V_{\text{eff}}(r) \to +\infty \qquad (r \to 0^+),$$

because of the angular-momentum term. At large radius,

$$V_{\text{eff}}(r) \to +\infty \qquad (r \to \infty),$$

because of the spring energy. Therefore the effective potential has at least one minimum for $r > 0$.

A circular relative motion occurs at $r = r_*$ satisfying

$$\frac{dV_{\text{eff}}}{dr} = 0.$$

Since

$$\frac{dV_{\text{eff}}}{dr} = K(r - r_0) - \frac{J^2}{\mu r^3},$$

the circular-radius condition is

$$\boxed{K(r_* - r_0) = \frac{J^2}{\mu r_*^3}.}$$

The second derivative is

$$\boxed{\frac{d^2V_{\text{eff}}}{dr^2} = K + \frac{3J^2}{\mu r^4} > 0.}$$

Therefore every circular relative motion allowed by this central spring potential is radially stable.

16.1 Special case: isotropic harmonic central force

If

$$r_0 = 0,$$

then

$$\boxed{V(r) = \frac{1}{2}Kr^2.}$$

The vector equation of motion becomes

$$\boxed{\mu\ddot{\mathbf{r}} = -K\mathbf{r}.}$$

Define

$$\omega^2 = \frac{K}{\mu}.$$

Then

$$\boxed{\ddot{\mathbf{r}} + \omega^2\mathbf{r} = \mathbf{0}.}$$

Each Cartesian component obeys the same harmonic-oscillator equation, so

$$\boxed{\mathbf{r}(t) = \mathbf{r}_0\cos(\omega t) + \frac{\mathbf{v}_0}{\omega}\sin(\omega t),}$$

where

$$\mathbf{r}_0 = \mathbf{r}(0), \qquad \mathbf{v}_0 = \dot{\mathbf{r}}(0).$$

This solution makes two facts immediate.

First, $\mathbf{r}(t)$ always remains in the plane spanned by $\mathbf{r}_0$ and $\mathbf{v}_0$, in agreement with angular-momentum conservation.

Second, the trajectory is built from two fixed vectors multiplied by $\cos(\omega t)$ and $\sin(\omega t)$. For nonzero angular momentum, this produces an ellipse centered on the force center; when the angular momentum vanishes, the ellipse degenerates to motion back and forth along a line. A detailed elimination of t is a useful exercise and will not be needed for the general central-force reduction.

17 What exactly has been reduced?

It is useful to collect the successive simplifications.

Start with two particles in three-dimensional space:

$$\boxed{(\mathbf{x}_1, \mathbf{x}_2) \quad \text{six configuration coordinates.}}$$

The coordinate change gives

$$(\mathbf{x}_1, \mathbf{x}_2) \longrightarrow (\mathbf{R}, \mathbf{r}).$$

Translation symmetry makes $\mathbf{R}$ free:

$$\boxed{M\ddot{\mathbf{R}} = \mathbf{0}.}$$

After removing this known overall translation, the interacting dynamics are carried by the three components of $\mathbf{r}$.

Rotational symmetry conserves $\mathbf{J}$, fixing the orbital plane:

$$\boxed{\mathbf{r}(t) \text{ lies in one plane.}}$$

The relative problem is therefore two-dimensional.

In polar coordinates, θ is cyclic:

$$\boxed{J = \mu r^2 \dot{\theta} = \text{constant}.}$$

Using that constant removes $\dot{\theta}$ from the radial equation:

$$\boxed{E_{\text{rel}} = \frac{1}{2}\mu \dot{r}^2 + V_{\text{eff}}(r).}$$

The only remaining dynamical coordinate in the reduced problem is r.

The full motion is then reconstructed in the reverse order:

$$\boxed{r(t) \longrightarrow \theta(t) \longrightarrow \mathbf{r}(t) \longrightarrow \mathbf{x}_1(t), \mathbf{x}_2(t).}$$

Reduction therefore does not mean throwing away the original system. It means solving the dynamics in variables that isolate the independent structure and then reconstructing the original description.

18 Summary

For two particles interacting through

$$V = V(|\mathbf{x}_2 - \mathbf{x}_1|),$$

define

$$\boxed{\mathbf{r} = \mathbf{x}_2 - \mathbf{x}_1, \qquad \mathbf{R} = \frac{m_1\mathbf{x}_1 + m_2\mathbf{x}_2}{M},}$$

with

$$\boxed{M = m_1 + m_2, \qquad \mu = \frac{m_1 m_2}{M}.}$$

The inverse transformation is

$$\boxed{\mathbf{x}_1 = \mathbf{R} - \frac{m_2}{M}\mathbf{r}, \qquad \mathbf{x}_2 = \mathbf{R} + \frac{m_1}{M}\mathbf{r}.}$$

The kinetic energy becomes

$$\boxed{T = \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2,}$$

and the Lagrangian separates:

$$\boxed{L = \frac{1}{2}M\dot{\mathbf{R}}^2 + \left[\frac{1}{2}\mu\dot{\mathbf{r}}^2 - V(r)\right].}$$

Translation symmetry gives

$$\boxed{\mathbf{P} = M\dot{\mathbf{R}} = \text{constant}.}$$

The relative equation is

$$\boxed{\mu\ddot{\mathbf{r}} = -\frac{dV}{dr}\hat{\mathbf{r}}.}$$

Rotational symmetry gives

$$\boxed{\mathbf{J} = \mathbf{r} \times \mu\dot{\mathbf{r}} = \text{constant},}$$

which confines the relative motion to a plane.

In polar coordinates,

$$\boxed{L_{\text{rel}} = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r),}$$

and the cyclic coordinate θ gives

$$\boxed{J = p_\theta = \mu r^2\dot{\theta} = \text{constant}.}$$

The areal velocity is therefore

$$\boxed{\dot{A} = \frac{J}{2\mu}.}$$

Finally,

$$\boxed{E_{\text{rel}} = \frac{1}{2}\mu\dot{r}^2 + V_{\text{eff}}(r),}$$

with

$$\boxed{V_{\text{eff}}(r) = V(r) + \frac{J^2}{2\mu r^2}.}$$

The effective potential determines allowed radial regions, turning points, circular orbits, and radial stability. The angular coordinate is reconstructed from

$$\boxed{\dot{\theta} = \frac{J}{\mu r^2}.}$$

The central conceptual chain is

interaction suggests $\mathbf{r}$

$\downarrow$

$(\mathbf{R}, \mathbf{r})$ separate translation and internal motion

$\downarrow$

translation symmetry $\rightarrow \mathbf{P}$

$\downarrow$

rotation symmetry $\rightarrow \mathbf{J}$

$\downarrow$

θ cyclic $\rightarrow J$

$\downarrow$

$V_{\text{eff}}(r)$ reduces the radial dynamics to one dimension.

In the next lecture, the same machinery will be specialized to the inverse-square potential

$$V(r) = -\frac{k}{r},$$

where the reduced dynamics lead to Keplerian orbit geometry and scattering.