

Graduate Classical Mechanics

Lecture 8: Coupled Oscillators as a Discrete Field

Normal modes, mode coordinates, symmetry, and mode counting

1 From many coordinates to a discrete field

Consider N scalar quantities collected into

$$\mathbf{q}(t) = \begin{pmatrix} q_1(t) \\ \vdots \\ q_N(t) \end{pmatrix}.$$

The vector of values by itself does *not* yet tell us that these degrees of freedom form a one-dimensional chain, a two-dimensional surface, or any other spatial object. We also need structure telling us which sites are local neighbors and how those local neighborhoods fit together.

There are two complementary ways to interpret the connection between a discrete field and a continuum field.

1.1 A discrete representation of an underlying continuum

One possibility is that a continuum object is primary and the discrete variables represent it. The values q_n might be point samples, local averages, or coefficients of basis functions. A particularly geometric choice is a simplicial piecewise-linear representation.

In one dimension, the sites are vertices on a line and neighboring nodal values define a linear interpolant on each segment. In two dimensions, vertices and triangular faces define a piecewise-planar field: the field is linear over each triangle and the nodal values determine the plane. In this viewpoint the mesh is a finite representation of a continuum object that we imagine already exists.

1.2 A continuum description emerging from discrete interactions

A second possibility is that the discrete objects are primary. No continuum is assumed in advance. Their interactions first define a graph, and the graph says which objects are local neighbors. A chain of local interactions has one-dimensional neighborhood structure.

For higher-dimensional systems we can use local subgraphs to identify the intended dimension and, when useful, add *virtual geometric edges* that do not represent new physical interactions but complete a dimension-preserving simplicial structure. Triangles can complete a two-dimensional simplicial manifold; tetrahedra can play the analogous role in three dimensions. The physical interaction graph and the geometric simplicial completion therefore need not be exactly the same object.

Edge differences are the first discrete analogue of local coordinate differences. For an oriented edge $i \rightarrow j$,

$$q_j - q_i$$

compares neighboring field values. Several independent edge directions in a local neighborhood can be used as a discrete chart. Overlapping neighborhoods provide the analogue of an atlas. To compare directions from one chart to the next we need a rule for transport. If parallel transport around a closed loop fails to return a direction to itself, that failure supplies a discrete notion of curvature:

$$\boxed{\text{parallel transport around a loop} \neq I \implies \text{curvature.}}$$

This connection/transport structure can be discussed before a metric is chosen. Metric data can then be added to assign lengths and angles; with a metric-compatible connection one obtains the discrete analogue of familiar Riemannian geometry.

The important lesson is

$$\boxed{\begin{array}{c} \text{field values } \mathbf{q} + \text{local incidence/interactions} + \text{simplicial structure} \\ + \text{transport} + \text{metric} \implies \text{discrete field geometry} \end{array}}$$

Thus the list $q_1, \dots, q_N$ is not the whole field description. The interaction/incidence structure provides the meaning of *local*; its simplicial completion can expose dimension and topology; transport and metric data add further geometry.

These two interpretations can become indistinguishable at long wavelength. Once a metric is supplied and neighboring sites are much closer than the scale over which the field changes appreciably, a fundamentally discrete interacting system and a discrete approximation to a continuum system can have the same effective description.

For the chain in this lecture, time is already continuous while the spatial label is discrete:

$$\boxed{q_n(t) = \text{discrete field value at site } n.}$$

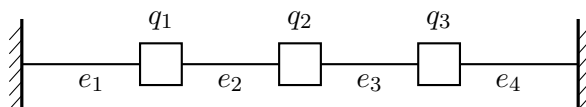
The next lecture will make the spatial description continuous as well, replacing the site label n by a continuous coordinate x and $q_n(t)$ by a field $\phi(x, t)$.

The central chain of ideas in this lecture is

$$\boxed{\begin{array}{l} \text{local connectivity} \rightarrow \text{spring extensions} \rightarrow A\mathbf{q} \rightarrow K = A^T A \\ \rightarrow \text{coupled equations} \rightarrow \text{eigenvectors} \rightarrow \text{normal modes} \rightarrow \text{mode coordinates.} \end{array}}$$

2 Build the mechanics from the spring extensions

Start with three moving masses between fixed endpoints. Let q_1, q_2, q_3 be displacements from equilibrium.



Choose the signed spring extensions

$$e_1 = q_1, \quad e_2 = q_2 - q_1, \quad e_3 = q_3 - q_2, \quad e_4 = -q_3.$$

These can be written in one matrix equation,

$$\boxed{\mathbf{e} = A\mathbf{q}}, \quad A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}.$$

Thus A maps quantities on nodes to differences on edges. This is the same node-to-edge structure that appeared at the beginning of the course.

For identical springs of spring constant κ ,

$$\begin{aligned} V &= \frac{\kappa}{2} \mathbf{e}^T \mathbf{e} \\ &= \frac{\kappa}{2} (A\mathbf{q})^T (A\mathbf{q}) \\ &= \frac{\kappa}{2} \mathbf{q}^T A^T A \mathbf{q}. \end{aligned}$$

Define the dimensionless stiffness operator

$$\boxed{K \equiv A^T A.}$$

Then

$$\boxed{V = \frac{\kappa}{2} \mathbf{q}^T K \mathbf{q}.$$

More generally, if the springs have a constitutive matrix C , the quadratic form is

$$\boxed{V = \frac{1}{2} \mathbf{q}^T A^T C A \mathbf{q}.$$

The present system has $C = \kappa I$, so we have factored the common κ outside and written $K = A^T A$.

2.1 The mass matrix

The kinetic energy has an analogous quadratic-form structure:

$$\boxed{T = \frac{1}{2} \dot{\mathbf{q}}^T M \dot{\mathbf{q}}.$$

For identical masses,

$$\boxed{M = mI,}$$

so

$$\dot{\mathbf{q}}^T M \dot{\mathbf{q}} = \dot{\mathbf{q}}^T (mI) \dot{\mathbf{q}} = m \dot{\mathbf{q}}^T \dot{\mathbf{q}}.$$

Thus

$$\boxed{T = \frac{m}{2} \dot{\mathbf{q}}^T \dot{\mathbf{q}}.$$

At the level of quadratic forms, the simplification is exactly the same statement one would write as $\mathbf{q}^T M \mathbf{q} = \mathbf{q}^T (mI) \mathbf{q} = m \mathbf{q}^T \mathbf{q}$. In the kinetic energy the relevant vector is $\dot{\mathbf{q}}$.

The Lagrangian is therefore

$$L = \frac{m}{2} \dot{\mathbf{q}}^T \dot{\mathbf{q}} - \frac{\kappa}{2} \mathbf{q}^T K \mathbf{q}.$$

The equal-mass chain is the especially transparent case $M = mI$. Later, when the masses are not all identical, M will become nontrivial and normal modes will lead to a generalized eigenvalue problem.

3 Equations of motion and the discrete second difference

Applying the Euler–Lagrange equations gives

$$m\ddot{\mathbf{q}} = -\kappa K \mathbf{q}.$$

For the fixed chain, an interior component is

$$m\ddot{q}_n = \kappa (q_{n+1} - 2q_n + q_{n-1}).$$

The spatial combination on the right is a second difference. At this stage we should regard it as a property of the discrete mechanics rather than immediately replacing it by a derivative.

The operator $K = A^T A$ has a useful physical interpretation:

$$\text{nodal displacement } \mathbf{q} \xrightarrow{A} \text{spring extensions } \mathbf{e} \xrightarrow{A^T} \text{net nodal restoring force.}$$

The tridiagonal matrix K is therefore not merely a convenient matrix pattern. It is the composition of two physical maps.

For the three-mass example,

$$K = A^T A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

4 A first MATLAB exploration at the command line

Before using a complete script, it is useful to reproduce the three-mass system interactively. Enter

```
A = [ 1  0  0;
      -1 1  0;
       0 -1 1;
       0  0 -1];

K = A'*A
```

Before MATLAB prints K , predict its pattern from the physical role of $A^T A$.

Next ask MATLAB for the eigensystem:

```
[Q,L] = eig(K);
diag(L)
Q
Q'*Q
Q'*K*Q
plot(1:3,Q,'o-')
```

Here L is being used as a short MATLAB name for the diagonal eigenvalue matrix Λ . The full lecture script uses the more explicit name `LambdaMatrix`.

The numerical experiment raises several mechanical questions. Why are there three eigenvectors? What do their shapes mean? Why does $Q^T K Q$ become diagonal? What do the corresponding eigenvalues have to do with frequencies? These questions lead directly to normal modes.

5 The eigenvalue equation and normal modes

Look for a motion in which the entire chain keeps one fixed spatial shape while only its overall amplitude changes:

$$\mathbf{q}(t) = \mathbf{u}_\alpha a_\alpha(t).$$

Substitute this into

$$m\ddot{\mathbf{q}} = -\kappa K \mathbf{q}.$$

We obtain

$$m\mathbf{u}_\alpha \ddot{a}_\alpha = -\kappa K \mathbf{u}_\alpha a_\alpha.$$

This becomes one scalar equation if the shape $\mathbf{u}_\alpha$ satisfies

$$K \mathbf{u}_\alpha = \lambda_\alpha \mathbf{u}_\alpha.$$

This is the *eigenvalue equation*. It says that $\mathbf{u}_\alpha$ is a special direction in configuration space: applying K does not turn the vector into a new direction. Instead,

$$K \mathbf{u}_\alpha \parallel \mathbf{u}_\alpha,$$

and K merely rescales the vector by λ_α .

For such a direction,

$$\ddot{a}_\alpha + \frac{\kappa}{m} \lambda_\alpha a_\alpha = 0.$$

Therefore

$$\omega_\alpha^2 = \frac{\kappa}{m} \lambda_\alpha.$$

The eigenvector gives the spatial shape and the eigenvalue gives the squared frequency, up to the factor κ/m .

This gives the physical meaning of a normal mode:

$$\text{normal mode} = \text{fixed spatial shape} \times \text{one scalar harmonic oscillation}.$$

Every mass oscillates with the same frequency. The components of $\mathbf{u}_\alpha$ determine their relative amplitudes and signs.

5.1 Exact fixed–fixed modes

For the fixed–fixed chain, the boundary conditions make standing sine waves plausible. Label the moving sites by $n = 1, \dots, N$ and the modes by $k = 1, \dots, N$. Try

$$u_n^{(k)} = \sqrt{\frac{2}{N+1}} \sin\left(\frac{nk\pi}{N+1}\right).$$

The fictitious endpoint values at $n = 0$ and $n = N + 1$ vanish automatically, exactly as fixed endpoints should.

Let

$$\theta_k = \frac{k\pi}{N+1}.$$

For an interior site,

$$(K\mathbf{u}^{(k)})_n = 2u_n^{(k)} - u_{n-1}^{(k)} - u_{n+1}^{(k)}.$$

Using

$$\sin((n+1)\theta) + \sin((n-1)\theta) = 2\cos\theta \sin(n\theta),$$

we find

$$K\mathbf{u}^{(k)} = \lambda_k \mathbf{u}^{(k)}, \quad \lambda_k = 2 \left[1 - \cos\left(\frac{k\pi}{N+1}\right) \right].$$

Equivalently,

$$\lambda_k = 4 \sin^2\left(\frac{k\pi}{2(N+1)}\right).$$

Thus the standing-wave guess is not merely qualitative: it gives the exact eigensystem of the fixed chain. The normalization $\sqrt{2/(N+1)}$ makes the eigenvectors orthonormal. For the three-mass example, it reduces to $1/\sqrt{2}$.

MATLAB can check the analytic result directly:

```
[Q,L] = eig(K);
[lambda,order] = sort(diag(L));
Q = Q(:,order);
L = diag(lambda);

n = (1:N)';
k = (1:N)';
Qexact = sqrt(2/(N+1))*sin(n*k'*pi/(N+1));
Lexact = diag(2*(1-cos(k*pi/(N+1))));

norm(K*Qexact - Qexact*Lexact)
norm(Qexact'*Qexact - eye(N))
norm(L-Lexact)
abs(Q'*Qexact)
```

The columns returned by `eig` may differ from the analytic columns by signs. The last line therefore compares the absolute overlaps.

6 Normal-mode coordinates decouple the system

Because K is real and symmetric, its eigenvectors can be chosen orthonormal. Collect them as columns of

$$Q = (\mathbf{u}_1 \quad \mathbf{u}_2 \quad \cdots \quad \mathbf{u}_N).$$

Then

$$\boxed{Q^T Q = I, \quad Q^T K Q = \Lambda,}$$

where

$$\Lambda = \text{diag}(\lambda_1, \dots, \lambda_N).$$

Now use the eigenvectors as a new basis for configuration space:

$$\boxed{\mathbf{q} = Q \mathbf{a}, \quad \mathbf{a} = Q^T \mathbf{q}.}$$

The components $a_\alpha(t)$ are the *normal-mode coordinates*. The transformation

$$\mathbf{a} = Q^T \mathbf{q}$$

is also a discrete spectral transform: it replaces site values by amplitudes of spatial basis functions. For the fixed-fixed chain, the matrix Q is the orthonormal discrete sine-transform basis (DST-I, up to conventional indexing and scaling choices). Thus the normal-mode decomposition is the fixed-boundary analogue of a discrete Fourier transform.

Substitution into the Lagrangian gives

$$\begin{aligned} L &= \frac{m}{2} \dot{\mathbf{a}}^T \dot{\mathbf{a}} - \frac{\kappa}{2} \mathbf{a}^T \Lambda \mathbf{a} \\ &= \sum_{\alpha=1}^N \left[\frac{m}{2} \dot{a}_\alpha^2 - \frac{\kappa \lambda_\alpha}{2} a_\alpha^2 \right]. \end{aligned}$$

Thus

$$\boxed{N \text{ coupled site coordinates} \longrightarrow N \text{ independent normal-mode coordinates.}}$$

Diagonalizing K is therefore not only a matrix operation. It is a physically useful change of generalized coordinates.

This also explains mode counting. An N -dimensional configuration space has N independent basis directions, and the orthonormal eigenvectors of K provide one especially useful basis. Hence there are N independent normal modes.

The more general problem

$$M \ddot{\mathbf{q}} + K_{\text{phys}} \mathbf{q} = 0$$

with nontrivial M leads to a generalized eigenvalue problem. We postpone that general theory until the later small-oscillation unit; the equal-mass chain makes the essential structure visible with minimal notation.

7 Close the chain into a ring: symmetry and a zero mode

Now connect the last site back to the first. Every spring still depends only on a difference of neighboring displacements. A uniform displacement therefore stretches no spring:

$$A\mathbf{1} = \mathbf{0}.$$

Consequently

$$K\mathbf{1} = A^T A\mathbf{1} = \mathbf{0}.$$

The uniform vector is an eigenvector with

$$\boxed{\lambda_0 = 0,} \quad \boxed{\omega_0 = 0.}$$

This zero mode can be predicted from symmetry without diagonalizing anything. The transformation

$$\boxed{q_n \longrightarrow q_n + c \quad \text{for every } n}$$

leaves every difference $q_{n+1} - q_n$ unchanged. Therefore it leaves the potential energy unchanged. This is a continuous translation in configuration space, exactly of the type studied in Lecture 7.

For this transformation the generator has $h_n = 1$ for every site, so the Noether quantity is

$$J = \sum_n p_n.$$

For equal masses,

$$\boxed{J = m \sum_n \dot{q}_n = \text{constant.}}$$

The same structure can therefore be read in several equivalent ways:

$$\boxed{\text{uniform displacement symmetry} \longleftrightarrow \text{no restoring force} \longleftrightarrow \lambda = 0 \longleftrightarrow \omega = 0.}$$

There is another symmetry of the ring that should not be confused with this one. Relabeling the sites by

$$n \longrightarrow n + 1$$

is a *discrete spatial translation*. It expresses the fact that every site has the same local environment. It does not itself give the Noether charge above, because the site shift is discrete rather than continuously connected to the identity. It does, however, explain why Fourier-like patterns are natural eigenmodes.

8 Fourier modes and mode counting on the ring

The fixed chain and the ring have the same local nearest-neighbor second-difference operator. What changes is the global boundary condition. For the fixed-fixed chain,

$$\boxed{k_j a = \frac{j\pi}{N+1}, \quad j = 1, \dots, N,}$$

and the natural real basis is the sine basis above. For a periodic ring,

$$k_r a = \frac{2\pi r}{N}, \quad r = 0, \dots, N-1.$$

The periodic eigenvectors are the complex Fourier basis, so the change from site coordinates to periodic mode amplitudes is a discrete Fourier transform (DFT). In both cases the same local operator has the eigenvalue relation

$$\lambda(k) = 2 - 2\cos(ka) = 4\sin^2\left(\frac{ka}{2}\right),$$

while the boundary condition chooses the allowed wave numbers and therefore the appropriate transform: a discrete sine transform for fixed endpoints and a DFT for periodic endpoints.

For the ring, try a complex spatial pattern

$$u_n = e^{ikna},$$

where a is the lattice spacing. Periodicity requires

$$u_{n+N} = u_n,$$

so

$$e^{ikNa} = 1.$$

Therefore the allowed wave numbers are

$$k_r = \frac{2\pi r}{Na}, \quad r = 0, 1, \dots, N-1.$$

There are exactly N allowed values, matching the N independent coordinates.

Apply the ring stiffness operator to this spatial pattern:

$$(Ku)_n = 2u_n - u_{n+1} - u_{n-1}.$$

Using $u_{n\pm 1} = u_n e^{\pm ika}$,

$$\begin{aligned} (Ku)_n &= \left(2 - e^{ika} - e^{-ika}\right) u_n \\ &= 4\sin^2\left(\frac{ka}{2}\right) u_n. \end{aligned}$$

Hence

$$\lambda(k) = 4\sin^2\left(\frac{ka}{2}\right),$$

and

$$\omega^2(k) = \frac{4\kappa}{m} \sin^2\left(\frac{ka}{2}\right).$$

This is the dispersion relation for the discrete nearest-neighbor chain. We will not explore the dispersion relation in depth here; the reference MATLAB script `waveq_reference.m` is retained for that purpose later.

The complex modes e^{ikna} are often the cleanest analytic basis. Since K is a real matrix, MATLAB's `eig` can instead return real sine/cosine combinations inside the degenerate $+k$ and $-k$ subspaces. These are different bases for the same eigenspaces, not different physics.

9 Using the complete MATLAB script

Once the matrix construction and normal-mode interpretation are understood, the full script simply automates the same steps for a larger chain. It can switch between fixed and periodic boundary conditions, constructs A , forms $K = A^T A$, diagonalizes K , evolves the equations of motion, and projects the simulated trajectory back into normal-mode coordinates.

The time integration uses a velocity-first symplectic-Euler update,

$$\mathbf{v}_{n+1} = \mathbf{v}_n - \frac{\kappa}{m} K \mathbf{q}_n \Delta t, \quad \mathbf{q}_{n+1} = \mathbf{q}_n + \mathbf{v}_{n+1} \Delta t.$$

This is closely related to the discrete update rules studied near the beginning of the course.

The most revealing diagnostic is the projection

$$\mathbf{a}(t) = Q^T \mathbf{q}(t).$$

A motion that looks complicated in site coordinates can become a small collection of clean oscillations in mode coordinates.

9.1 Complete MATLAB script

The following is the complete `Lecture08_modes.m` used in lecture. It is included so that you can copy it, change parameters and initial conditions, switch the boundary condition, and experiment.

```
1 %% Lecture 8: coupled oscillators as a discrete field
2 % Build the spatial difference matrix A, form K=A'*A, find normal modes,
3 % and evolve the chain with a symplectic-Euler update.
4 %
5 % Two boundary conditions are available:
6 % 'fixed' : N moving masses between fixed endpoints
7 % 'periodic' : N masses on a ring
8 %
9 % The point of this version is the chain -> K=A'*A -> normal modes story.
10 % The older waveq.m is kept separately as a reference for FFT and
11 % dispersion-relation plots.
12
13 clear; close all; clc;
14
15 %% physical parameters
16 N = 22; % number of moving masses / field samples
17 m = 1; % mass at each site
18 kappa = 1; % spring constant
19 boundary = 'fixed'; % 'fixed' or 'periodic'
20
21 %% time integration
22 TT = 40; % total simulation time
23 plotskip = 5; % update animation every plotskip time steps
24
25 %% build the discrete difference/incidence matrix A
26 switch lower(boundary)
27     case 'fixed'
28         % q_0=q_{N+1}=0. The N+1 spring extensions are A*q.
29         A = zeros(N+1,N);
```

```

30     A(1,1) = 1;
31     for e = 2:N
32         A(e,e-1) = -1;
33         A(e,e) = 1;
34     end
35     A(N+1,N) = -1;
36
37     case 'periodic'
38         % Ring: row e contains  $q_{e+1}-q_e$ , with  $q_{N+1}=q_1$ .
39         A = zeros(N,N);
40         for e = 1:N-1
41             A(e,e) = -1;
42             A(e,e+1) = 1;
43         end
44         A(N,N) = -1;
45         A(N,1) = 1;
46
47     otherwise
48         error('boundary must be ''fixed'' or ''periodic''.')
49 end
50
51 %% stiffness operator and normal modes
52 K = A'*A;
53 [Q,LambdaMatrix] = eig(K);
54 Lambda = diag(LambdaMatrix);
55 [Lambda,order] = sort(real(Lambda));
56 Q = real(Q(:,order));
57 omega = sqrt((kappa/m)*max(Lambda,0));
58
59 % Choose dt from the highest normal-mode frequency.
60 omegaMax = max(omega);
61 if omegaMax > 0
62     dt = 0.10/omegaMax;
63 else
64     dt = 0.01;
65 end
66 Nt = floor(TT/dt) + 1;
67 t = (0:Nt-1)*dt;
68
69 %% inspect  $K=A^T A$ 
70 figure(1); clf;
71 imagesc(K); axis image; colorbar;
72 xlabel('site'); ylabel('site');
73 title(['K=A^T A, boundary = ',boundary]);
74 drawnow;
75
76 %% inspect several normal-mode shapes
77 nshow = min(6,N);
78 figure(2); clf;
79 for a = 1:nshow
80     subplot(nshow,1,a);
81     plot(1:N,Q(:,a),'o-');
82     ylabel(sprintf('u_%d',a));
83     if a == 1
84         title('First normal-mode shapes');
85     end
86 end
87 xlabel('site n');
88 drawnow;

```

```

89
90 %% initial condition: superposition of a few normal modes
91 % Change these lists to excite different modes.
92 if strcmpi(boundary,'periodic')
93     % Skip the zero-frequency uniform mode by default.
94     modeList = unique(min([2 4 7],N));
95 else
96     modeList = unique(min([1 3 6],N));
97 end
98 ampList = [1.0 0.60 0.35];
99 ampList = ampList(1:numel(modeList));
100
101 q = zeros(N,1);
102 for j = 1:numel(modeList)
103     q = q + ampList(j)*Q(:,modeList(j));
104 end
105 v = zeros(N,1);
106
107 % Normalize only for plotting convenience.
108 q = q/max(abs(q));
109
110 %% save the trajectory
111 qs = zeros(N,Nt);
112 vs = zeros(N,Nt);
113 qs(:,1) = q;
114 vs(:,1) = v;
115 qmax = 1.1*max(abs(q));
116
117 %% evolve  $m\ddot{q} = -\kappa K q$ 
118 % Symplectic Euler: update v first, then q.
119 figure(3); clf;
120 for nt = 2:Nt
121     v = v - (kappa/m)*(K*q)*dt;
122     q = q + v*dt;
123
124     qs(:,nt) = q;
125     vs(:,nt) = v;
126
127     if mod(nt,plotskip) == 0
128         plot(1:N,q,'o-','LineWidth',1.2);
129         xlabel('site n'); ylabel('q_n(t)');
130         title(sprintf('Coupled chain, t = %.2f',t(nt)));
131         axis([1 N -qmax qmax]);
132         grid on;
133         drawnow;
134     end
135 end
136
137 %% space-time view of the discrete field  $q_n(t)$ 
138 figure(4); clf;
139 imagesc(t,1:N,qs);
140 axis xy; colorbar;
141 xlabel('time'); ylabel('site n');
142 title('Discrete field  $q_n(t)$ ');
143
144 %% project the motion onto the normal modes
145 % Since Q is orthonormal for this equal-mass problem,  $a(t)=Q^T q(t)$ .
146 a = Q'*qs;
147

```

```

148 figure(5); clf;
149 imagesc(t,1:N,a);
150 axis xy; colorbar;
151 xlabel('time'); ylabel('mode index');
152 title('Normal-mode amplitudes  $a_{\alpha}(t) = u_{\alpha}^T q(t)$ ');
153
154 %% eigenfrequency spectrum
155 figure(6); clf;
156 plot(1:N,omega,'o-');
157 xlabel('mode index'); ylabel('\omega_{\alpha}');
158 title('Normal-mode frequencies');
159 grid on;
160
161 %% useful checks printed to the command window
162 fprintf('\nLecture 8 coupled-chain calculation\n');
163 fprintf('boundary: %s\n',boundary);
164 fprintf('N = %d, number of normal modes = %d\n',N,numel(omega));
165 fprintf('||K-A''A|| = %.3e\n',norm(K-A'*A));
166
167 if strcmpi(boundary,'periodic')
168     uniform = ones(N,1)/sqrt(N);
169     fprintf('periodic zero-mode check ||K*1|| = %.3e\n',norm(K*uniform));
170     fprintf('smallest omega = %.6g\n',omega(1));
171 end

```

Useful experiments include changing N , exciting only one eigenvector, superposing several modes, changing from fixed to periodic, and deliberately adding the periodic zero mode to the initial condition.

10 The continuum is already visible

Return to the interior equation

$$m\ddot{q}_n = \kappa (q_{n+1} - 2q_n + q_{n-1}).$$

If neighboring sites are separated by a distance a , then for a slowly varying spatial pattern

$$\frac{q_{n+1} - 2q_n + q_{n-1}}{a^2} \longrightarrow \frac{\partial^2 \phi}{\partial x^2}.$$

Thus the discrete equation has the structural form

$$\ddot{q}_n \sim \frac{\kappa a^2}{m} \frac{q_{n+1} - 2q_n + q_{n-1}}{a^2},$$

which previews

$$\frac{\partial^2 \phi}{\partial t^2} \propto \frac{\partial^2 \phi}{\partial x^2}.$$

We are not yet taking a careful physical continuum limit; in a genuine continuum limit the discrete parameters must scale appropriately with the lattice spacing. The important observation here is structural: a spatial second difference is becoming a spatial second derivative.

The field description is therefore already latent in the discrete mechanics:

$$\boxed{q_n(t) \longrightarrow \phi(x,t).}$$

The next lecture will ask the corresponding variational question: what becomes of the Lagrangian and the action when the discrete spatial sum becomes an integral over space?

11 Summary

For a mass–spring chain,

$$\mathbf{e} = A\mathbf{q}, \quad V = \frac{1}{2}\mathbf{q}^T A^T C A \mathbf{q}, \quad T = \frac{1}{2}\dot{\mathbf{q}}^T M \dot{\mathbf{q}}.$$

For identical masses and springs,

$$M = mI, \quad C = \kappa I, \quad K = A^T A,$$

and the equations of motion are

$$m\ddot{\mathbf{q}} = -\kappa K \mathbf{q}.$$

The eigenvalue equation

$$K \mathbf{u}_\alpha = \lambda_\alpha \mathbf{u}_\alpha$$

identifies special directions in configuration space for which $K \mathbf{u}_\alpha$ is parallel to $\mathbf{u}_\alpha$. Each such direction gives a normal mode with

$$\omega_\alpha^2 = \frac{\kappa}{m} \lambda_\alpha.$$

Collecting the eigenvectors into Q defines the normal coordinates

$$\mathbf{a} = Q^T \mathbf{q},$$

which turn the coupled system into independent oscillators.

The field values alone do not specify a discrete field geometry: locality comes from the incidence/interaction structure, while simplicial completion, transport, and metric data can add dimension, topology, curvature, lengths, and angles.

For fixed endpoints, the exact eigenvectors form an orthonormal sine basis, so the normal-mode transform is a discrete sine transform. For a ring, uniform displacement is a continuous symmetry and therefore produces a zero-frequency mode; the discrete translation symmetry makes the periodic Fourier basis natural, and the mode transform becomes a DFT. The fixed and periodic systems share the same local eigenvalue relation $\lambda(k) = 2 - 2 \cos(ka)$ but differ in their allowed wave numbers. Finally, the discrete second difference points directly toward the spatial derivatives of continuum field theory.