

Graduate Classical Mechanics

Lecture 7: Constraint Forces, Continuous Symmetry, and Noether's Theorem

From recovering reaction forces to conserved quantities generated by symmetry

1 Return to the Lecture 6 pendulum

Lecture 6 used a coordinate that built the pendulum constraint into the description from the beginning. With

$$x = \ell_0 \sin \theta, \quad y = -\ell_0 \cos \theta,$$

the Lagrangian was

$$L = \frac{1}{2} m \ell_0^2 \dot{\theta}^2 + mg \ell_0 \cos \theta,$$

and the Euler–Lagrange equation gave

$$\frac{d}{dt} (m \ell_0^2 \dot{\theta}) = -mg \ell_0 \sin \theta,$$

or

$$\ddot{\theta} + \frac{g}{\ell_0} \sin \theta = 0.$$

The tension does not appear in this reduced equation. That does not mean the tension vanished physically. The coordinate θ parameterizes the allowed tangent motion, while the radial reaction force has been projected out.

This lecture begins with the complementary question:

How can we keep the constraint explicitly and recover the reaction force that enforces it?

The answer will lead naturally to Lagrange multipliers. After that, we return to the Lecture 6 observation that a cyclic coordinate gives a conserved momentum and ask whether there is a deeper symmetry principle behind it.

2 Constraint surfaces, generalized coordinates, and time

Let

$$\mathbf{q} = (q_1, \dots, q_n)^T$$

be the coordinates we retain. At this stage they need not yet form a minimal independent set. One holonomic constraint is written

$$F(\mathbf{q}, t) = 0.$$

Cartesian coordinates are one possible choice of $\mathbf{q}$, but they are not required. If instead we choose independent generalized coordinates that parameterize only the allowed configurations, then the

constraint is satisfied identically and the corresponding reaction force is projected out, as in Lecture 6.

There are two different kinds of change that must be kept distinct.

2.1 Actual motion

Along the actual trajectory, the constraint remains satisfied, so

$$\dot{F} = \frac{\partial F}{\partial \mathbf{q}} \cdot \dot{\mathbf{q}} + \frac{\partial F}{\partial t} = 0.$$

If the constraint depends explicitly on time, then

$$\frac{\partial F}{\partial \mathbf{q}} \cdot \dot{\mathbf{q}} = -\frac{\partial F}{\partial t}.$$

Thus the actual velocity need not be tangent to the instantaneous constraint surface when that surface itself moves with time.

2.2 Virtual displacement

A virtual displacement compares nearby allowed configurations at the *same time*. Therefore

$$\delta t = 0,$$

and

$$\delta F = \frac{\partial F}{\partial \mathbf{q}} \cdot \delta \mathbf{q} = 0.$$

Consequently, at a fixed time t_0 , the vector

$$\frac{\partial F}{\partial \mathbf{q}}$$

is normal to the instantaneous constraint surface $F(\mathbf{q}, t_0) = 0$, while the allowed virtual displacements $\delta \mathbf{q}$ are tangent to it.

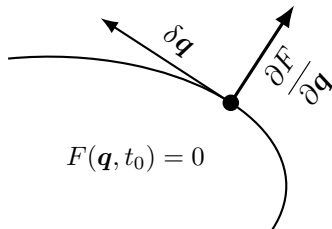


Figure 1: At fixed time, allowed virtual displacements are tangent to the instantaneous constraint surface, while $\partial F / \partial \mathbf{q}$ is normal.

The normal direction in Fig. 1 contains only the derivative with respect to the configuration coordinates. The explicit derivative $\partial F / \partial t$ describes motion of the constraint surface in time; it is not part of the fixed-time normal vector.

3 Ideal reaction force and the choice of constraint function

Lecture 6 allowed generalized nonconservative forces on the right-hand side of Lagrange's equation. The particular generalized force of interest here is the reaction force. We write

$$\mathbf{R} \equiv \mathbf{Q}^{(R)}.$$

For an ideal constraint, the reaction does no virtual work:

$$\mathbf{R} \cdot \delta \mathbf{q} = 0$$

for every allowed virtual displacement. Because the allowed variations are tangent to the constraint surface, the generalized reaction must lie in its normal direction. Hence, for one constraint,

$$\mathbf{R} = \lambda \frac{\partial F}{\partial \mathbf{q}}.$$

The vector $\mathbf{R}$ is a generalized reaction-force vector. Its components are conjugate to the chosen generalized coordinates and therefore need not all have ordinary force units. The multiplier λ measures the magnitude only relative to the particular function F used to describe the constraint.

3.1 The physical constraint is the surface, not the formula

For the pendulum, the physical condition is simply

$$\ell = \ell_0.$$

Many formulas can have this as their zero set, but not all are equally useful in the multiplier method. The defining function should be differentiable and regular on the constraint surface:

$$\frac{\partial F}{\partial \mathbf{q}} \neq 0 \quad \text{on } F = 0.$$

For example,

$F(\ell)$	$\frac{\partial F}{\partial \ell}$ at ℓ_0	multiplier use
$\ell - \ell_0$	1	good
$(\ell - \ell_0)^2$	0	bad: no normal direction there
$ \ell - \ell_0 $	undefined	bad: not differentiable
$\ell^2 - \ell_0^2$	$2\ell_0$	good locally if $\ell_0 > 0$

The function $\ell^2 - \ell_0^2$ also has the zero $\ell = -\ell_0$, but if ℓ is a radial length with $\ell \geq 0$, that extra branch is excluded.

More generally, suppose

$$\tilde{F} = g(F), \quad g(0) = 0, \quad g'(0) \neq 0.$$

Then on the constraint surface,

$$\frac{\partial \tilde{F}}{\partial \mathbf{q}} = g'(0) \frac{\partial F}{\partial \mathbf{q}}.$$

The normal direction is unchanged, only its normalization changes. Therefore

$$\mathbf{R} = \lambda \frac{\partial F}{\partial \mathbf{q}} = \tilde{\lambda} \frac{\partial \tilde{F}}{\partial \mathbf{q}}.$$

The multiplier itself changes when the defining function is rescaled, while the physical generalized reaction $\mathbf{R}$ does not.

4 Discovering the augmented Lagrangian

The multiplier term should not appear as an unexplained recipe. It follows directly from the reaction-force form just derived.

With one constraint, the forced Euler–Lagrange equation is

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\mathbf{q}}} \right) - \frac{\partial L}{\partial \mathbf{q}} = \lambda \frac{\partial F}{\partial \mathbf{q}}.$$

Move the reaction term to the left:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\mathbf{q}}} \right) - \frac{\partial L}{\partial \mathbf{q}} - \lambda \frac{\partial F}{\partial \mathbf{q}} = 0.$$

Now notice that, because $\lambda(t)$ is an independent multiplier variable,

$$-\frac{\partial L}{\partial \mathbf{q}} - \lambda \frac{\partial F}{\partial \mathbf{q}} = -\frac{\partial}{\partial \mathbf{q}} (L + \lambda F).$$

A holonomic constraint $F(\mathbf{q}, t) = 0$ has no $\dot{\mathbf{q}}$ dependence, so

$$\frac{\partial}{\partial \dot{\mathbf{q}}} (L + \lambda F) = \frac{\partial L}{\partial \dot{\mathbf{q}}}.$$

Therefore the whole equation can be written

$$\frac{d}{dt} \left[\frac{\partial}{\partial \dot{\mathbf{q}}} (L + \lambda F) \right] - \frac{\partial}{\partial \mathbf{q}} (L + \lambda F) = 0.$$

Only now do we define the augmented Lagrangian

$$L_\lambda \equiv L + \lambda F.$$

Variation with respect to λ gives

$$F(\mathbf{q}, t) = 0.$$

Thus $L + \lambda F$ does not invent the constraint force. Geometry and virtual work first tell us that the reaction has the form $\lambda \partial F / \partial \mathbf{q}$; the augmented Lagrangian packages that already-understood force into the variational formalism.

For several independent constraints $F_\alpha(\mathbf{q}, t) = 0$, the corresponding generalization is

$$\mathbf{R} = \sum_{\alpha} \lambda_{\alpha} \frac{\partial F_{\alpha}}{\partial \mathbf{q}}, \quad L_{\lambda} = L + \sum_{\alpha} \lambda_{\alpha} F_{\alpha}.$$

5 Pendulum example: mirror the whole construction

Return to the same pendulum, but do *not* impose the fixed length at the beginning. Retain

$$\mathbf{q} = \begin{pmatrix} \ell \\ \theta \end{pmatrix},$$

and impose

$$F(\ell, \theta) = \ell - \ell_0 = 0.$$

Then

$$\frac{\partial F}{\partial \mathbf{q}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

so the generalized reaction is

$$\mathbf{R} = \lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Before doing any dynamics, we already know the structure: the reaction appears only in the radial equation and not in the angular equation. With this normalization, λ itself is the signed radial generalized reaction force.

Using

$$x = \ell \sin \theta, \quad y = -\ell \cos \theta,$$

we obtain

$$T = \frac{1}{2}m \left(\dot{\ell}^2 + \ell^2 \dot{\theta}^2 \right), \quad V = -mg\ell \cos \theta,$$

and therefore

$$L = \frac{1}{2}m \left(\dot{\ell}^2 + \ell^2 \dot{\theta}^2 \right) + mg\ell \cos \theta.$$

The augmented Lagrangian is

$$L_\lambda = L + \lambda(\ell - \ell_0).$$

5.1 Angular equation: recover Lecture 6

The multiplier does not enter the θ equation because

$$\frac{\partial F}{\partial \theta} = 0.$$

The Euler–Lagrange equation gives

$$\frac{d}{dt} \left(m\ell^2 \dot{\theta} \right) = -mg\ell \sin \theta.$$

On the constraint,

$$\ell = \ell_0, \quad \dot{\ell} = 0,$$

so

$$\ddot{\theta} + \frac{g}{\ell_0} \sin \theta = 0,$$

exactly the Lecture 6 equation.

5.2 Radial equation: recover the missing reaction

The ℓ equation is

$$m\ddot{\ell} - m\ell\dot{\theta}^2 - mg \cos \theta = \lambda.$$

On the constraint,

$$\ell = \ell_0, \quad \dot{\ell} = 0, \quad \ddot{\ell} = 0,$$

so

$$\lambda = -m\ell_0\dot{\theta}^2 - mg \cos \theta.$$

Positive ℓ points outward, so the inward radial reaction is

$$R_{\text{inward}} = -\lambda = m\ell_0\dot{\theta}^2 + mg \cos \theta.$$

For a taut string this is the tension. For a rigid rod it is the signed inward radial constraint force.

The two descriptions are therefore complementary:

$\ell = \ell_0$ built into the coordinates $\longrightarrow$ one coordinate θ , reaction disappears, ℓ retained and $F = 0$ $\longrightarrow$ two coordinates + multiplier, reaction recovered.
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These are not competing methods. They are two descriptions of the same constrained mechanics.

5.3 The same calculation could be done in Cartesian coordinates

One could instead retain (x, y) and impose, for example,

$$F(x, y) = \frac{1}{2} (x^2 + y^2 - \ell_0^2) = 0.$$

Then

$$\frac{\partial F}{\partial \mathbf{r}} = \begin{pmatrix} x \\ y \end{pmatrix},$$

and the multiplier force is radial. Projecting the resulting equations tangent to the circle reproduces the θ equation, while projecting radially recovers the tension. The (ℓ, θ) formulation is preferable here because it makes the comparison with Lecture 6 immediate.

6 Return to the cyclic-coordinate result

From here on, unless stated otherwise, let q_j denote independent generalized coordinates. Define

$$p_j \equiv \frac{\partial L}{\partial \dot{q}_j},$$

so the Euler–Lagrange equation reads

$$\dot{p}_j = \frac{\partial L}{\partial q_j}.$$

Lecture 6 ended with the special case

$$\frac{\partial L}{\partial q_k} = 0 \implies \boxed{p_k = \text{constant.}}$$

A coordinate satisfying $\partial L / \partial q_k = 0$ is called *cyclic* or *ignorable*.

What does this mean geometrically? If L does not depend on q_k , then the shift

$$\boxed{q_k \longrightarrow q_k + \epsilon}$$

leaves L unchanged. Thus a cyclic coordinate already describes a continuous translation symmetry in generalized-coordinate space. The coordinate q_k might be an ordinary Cartesian position, an angle, or something less familiar.

7 A conserved quantity can be hidden by the coordinates

Consider two particles moving in one dimension with an interaction depending only on their separation:

$$\boxed{L = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - V(x_1 - x_2).}$$

Neither x_1 nor x_2 is cyclic separately. Nevertheless, translate both particles by the same amount:

$$\boxed{x_1 \longrightarrow x_1 + \epsilon, \quad x_2 \longrightarrow x_2 + \epsilon.}$$

The separation and the velocities are unchanged, so the Lagrangian is unchanged.

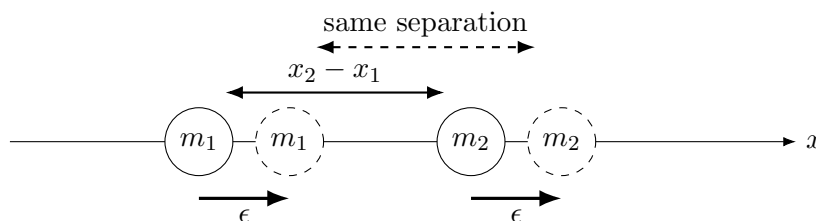


Figure 2: A common translation changes the absolute positions of both particles but leaves their separation unchanged.

This motivates the broader question:

If conservation can occur even when no single original coordinate is absent from L , is there a deeper structure than “cyclic coordinate”?

The answer is continuous symmetry.

7.1 Coordinates can make the hidden symmetry manifest

Define the center-of-mass and relative coordinates

$$X = \frac{m_1 x_1 + m_2 x_2}{M}, \quad r = x_1 - x_2, \quad M = m_1 + m_2,$$

and the reduced mass

$$\mu = \frac{m_1 m_2}{M}.$$

Then

$$L = \frac{1}{2} M \dot{X}^2 + \frac{1}{2} \mu \dot{r}^2 - V(r).$$

Now X is visibly cyclic, and

$$p_X = M \dot{X} = m_1 \dot{x}_1 + m_2 \dot{x}_2 = P_{\text{tot}}.$$

The symmetry was present in the original (x_1, x_2) coordinates; the change to (X, r) merely made it manifest.

8 Continuous and discrete symmetry

A symmetry is a change in the description that leaves the relevant physical structure unchanged. Rotating a circle about its center leaves the circle unchanged. A square or an equilateral triangle also has rotational symmetry, but only for special finite angles.

The distinction matters for Noether's theorem:

continuous symmetry = a symmetry connected to the identity by arbitrarily small changes.

A circle has continuous rotational symmetry. A square or an equilateral triangle has only discrete rotational symmetry. In the continuous case, the symmetry can be generated by an infinitesimal displacement at each point in configuration space.

At fixed time, write a small transformation as

$$q'_n = q_n + \epsilon h_n(\mathbf{q}, t),$$

with

$$\delta_s q_n = \epsilon h_n(\mathbf{q}, t), \quad \delta_s t = 0.$$

The functions h_n specify the scalar components of the infinitesimal symmetry displacement. Collectively they form a vector field $\mathbf{h}(\mathbf{q}, t)$ on configuration space.

8.1 Flattened components and particle blocks

There are two useful ways to organize the same configuration-space vector. In the generic Noether derivation we use a flattened scalar index,

$$\mathbf{q} = (q_1, \dots, q_{dN})^T, \quad \mathbf{h} = (h_1, \dots, h_{dN})^T,$$

so q_n and h_n are scalar components.

For N particles in d dimensions, the same vectors can instead be grouped into N blocks of dimension d :

$$\mathbf{q} = \begin{pmatrix} q_1 \\ \vdots \\ q_N \end{pmatrix}, \quad \mathbf{h} = \begin{pmatrix} h_1 \\ \vdots \\ h_N \end{pmatrix}, \quad \mathbf{q}_i, \mathbf{h}_i \in \mathbb{R}^d.$$

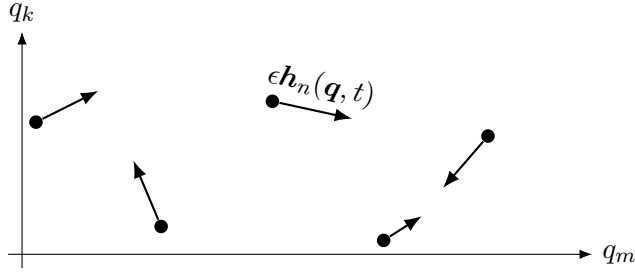
Thus

$$\delta_s q_n = \epsilon h_n$$

is the scalar-component form, while in particle-block notation the same transformation is

$$\delta_s \mathbf{q}_i = \epsilon \mathbf{h}_i.$$

Later, when the blocks are ordinary particle positions, we write $\mathbf{q}_i = \mathbf{r}_i$. Then spatial translation has $\mathbf{h}_i = \mathbf{a}$, and rotation has $\mathbf{h}_i = \hat{\mathbf{n}} \times \mathbf{r}_i$. No new transformation is being introduced; only the indexing has been regrouped.



An infinitesimal transformation field on configuration space.

Figure 3: The generator $\mathbf{h}(\mathbf{q}, t)$ assigns an infinitesimal displacement direction to each configuration. The drawing shows the projection of this generally dN -dimensional vector field onto the (q_k, q_m) plane.

The generator may depend explicitly on time even though the transformation itself is taken at fixed time. The condition $\delta_s t = 0$ means that we compare configurations at the same value of the independent variable t . Time translation changes t itself and will be treated separately at the end of the lecture.

9 Action viewpoint: the boundary term becomes the conserved quantity

The action viewpoint gives a direct reason that a continuous symmetry produces a conserved quantity along a physical trajectory.

In the derivation of Hamilton's principle, we considered arbitrary path variations that vanished at the endpoints. A symmetry transformation is different: it moves the entire trajectory, including the endpoint configurations.

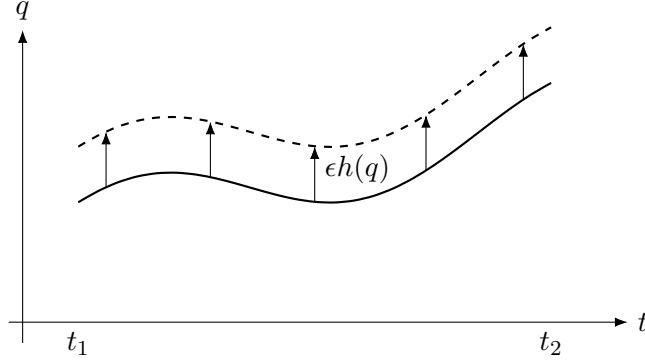


Figure 4: A fixed-time symmetry transformation moves the entire trajectory, including the endpoint configurations. The arrows are vertical because $\delta_s t = 0$.

Before integrating by parts, the first-order change in the action is

$$\delta_s \mathcal{S} = \int_{t_1}^{t_2} \sum_n \left[\frac{\partial L}{\partial q_n} \delta_s q_n + \frac{\partial L}{\partial \dot{q}_n} \delta_s \dot{q}_n \right] dt.$$

Integrating the second term by parts gives

$$\delta_s \mathcal{S} = \int_{t_1}^{t_2} \sum_n \left[\frac{\partial L}{\partial q_n} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_n} \right) \right] \delta_s q_n dt + \left[\sum_n p_n \delta_s q_n \right]_{t_1}^{t_2}.$$

When deriving Euler–Lagrange in the earlier variational lectures, fixed endpoints made the last term vanish. Here we do *not* impose $\delta_s q_n(t_1) = \delta_s q_n(t_2) = 0$ because the symmetry moves the endpoints.

On a physical trajectory, the Euler–Lagrange equations make the bulk integral vanish. Therefore

$$\delta_s \mathcal{S} = \left[\sum_n p_n \delta_s q_n \right]_{t_1}^{t_2}.$$

Using

$$\delta_s q_n = \epsilon h_n,$$

we obtain

$$\delta_s \mathcal{S} = \epsilon \left[\sum_n p_n h_n \right]_{t_1}^{t_2}.$$

If the transformation is an exact symmetry of the action,

$$\delta_s \mathcal{S} = 0,$$

then define

$$J(t) \equiv \sum_n p_n(t) h_n(\mathbf{q}(t), t),$$

and conclude

$$J(t_2) - J(t_1) = 0.$$

Because the same symmetry argument applies to any interval $[t_1, t_2]$ along the physical trajectory,

$$\boxed{J = \text{constant.}}$$

This shows the logical structure precisely:

$$\boxed{\underbrace{\text{equations of motion}}_{\text{bulk term}=0} + \underbrace{\text{continuous action symmetry}}_{\delta_s \mathcal{S}=0} \implies \underbrace{J(t_2) = J(t_1)}_{\text{conservation}}.}$$

Symmetry by itself is not the conservation law. The conservation statement uses both the symmetry and the dynamics.

10 First checks: cyclic coordinate and spatial translation

10.1 Cyclic coordinate

For

$$q_k \longrightarrow q_k + \epsilon,$$

the generator has

$$h_k = 1, \quad h_{n \neq k} = 0.$$

Thus

$$\boxed{J = p_k.}$$

The cyclic-coordinate result from Lecture 6 is therefore the simplest Noether example.

10.2 Spatial translation

Translate every particle by the same constant direction $\mathbf{a}$:

$$\boxed{\mathbf{r}_i \longrightarrow \mathbf{r}_i + \epsilon \mathbf{a} .}$$

The generator is

$$\boxed{\mathbf{h}_i = \mathbf{a} .}$$

The Noether quantity is

$$J = \sum_i \mathbf{p}_i \cdot \mathbf{a} = \mathbf{a} \cdot \sum_i \mathbf{p}_i .$$

If the action is invariant under translations in the direction $\mathbf{a}$, then

$$\boxed{\mathbf{a} \cdot \mathbf{P} = \text{constant}, \quad \mathbf{P} \equiv \sum_i \mathbf{p}_i .}$$

If all spatial translations are symmetries, then

$$\boxed{\mathbf{P} = \text{constant.}}$$

For ordinary kinetic energy, $\mathbf{p}_i = m_i \dot{\mathbf{r}}_i$, so $\mathbf{P}$ is the familiar total linear momentum.

For the two-particle example, the common translation leaves $x_1 - x_2$ unchanged, so the interaction and the action are unchanged. The equations of motion then turn this symmetry into conservation of total momentum.

More generally, if an interaction depends only on relative positions such as $\mathbf{r}_i - \mathbf{r}_j$, a common translation leaves all of those differences unchanged. This is the structural origin of translation symmetry in many isolated particle systems.

11 Rotation: build the infinitesimal generator

Start in two dimensions with

$$\mathbf{r}(\theta) = \begin{pmatrix} r \cos \theta \\ r \sin \theta \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

For a small rotation $\theta \rightarrow \theta + \epsilon$,

$$\mathbf{r}(\theta + \epsilon) - \mathbf{r}(\theta) = \epsilon \frac{d\mathbf{r}}{d\theta} + O(\epsilon^2).$$

Therefore the infinitesimal symmetry displacement is

$$\delta_s \mathbf{r} = \epsilon \frac{d\mathbf{r}}{d\theta} = \epsilon \begin{pmatrix} -r \sin \theta \\ r \cos \theta \end{pmatrix} = \epsilon \begin{pmatrix} -y \\ x \end{pmatrix}.$$

Thus

$$\boxed{\delta_s x = -\epsilon y,} \quad \boxed{\delta_s y = \epsilon x.}$$

The infinitesimal displacement is tangent to the circular orbit generated by rotation.

In the geometric-algebra notation introduced earlier, with

$$I = \hat{x}\hat{y},$$

we have

$$\mathbf{r}I = (x\hat{x} + y\hat{y})\hat{x}\hat{y} = x\hat{y} - y\hat{x},$$

so

$$\boxed{\delta_s \mathbf{r} = \epsilon \mathbf{r}I.}$$

In ordinary three-vector language, the same planar rotation is

$$\boxed{\delta_s \mathbf{r} = \epsilon \hat{z} \times \mathbf{r}.}$$

For rotation about a general axis $\hat{\mathbf{n}}$,

$$\boxed{\mathbf{h}_i = \hat{\mathbf{n}} \times \mathbf{r}_i.}$$

The Noether quantity is therefore

$$J = \sum_i \mathbf{p}_i \cdot (\hat{\mathbf{n}} \times \mathbf{r}_i).$$

Using the scalar triple-product identity,

$$J = \hat{\mathbf{n}} \cdot \sum_i (\mathbf{r}_i \times \mathbf{p}_i).$$

Define

$$\mathbf{L} \equiv \sum_i \mathbf{r}_i \times \mathbf{p}_i.$$

Rotational symmetry about $\hat{\mathbf{n}}$ gives

$$\hat{\mathbf{n}} \cdot \mathbf{L} = \text{constant.}$$

If the system is invariant under arbitrary spatial rotations, then

$$\mathbf{L} = \text{constant.}$$

Again, rotational symmetry by itself is not the conservation law. Rotational invariance of the action together with the equations of motion gives conservation of the corresponding component of angular momentum.

12 Local differential form: the same theorem point by point

The global action argument can be compressed into a local differential statement. We continue to hold time fixed under the symmetry transformation:

$$\delta_s t = 0.$$

The generator $h_n(\mathbf{q}, t)$ may nevertheless depend explicitly on time.

With

$$\delta_s q_n = \epsilon h_n,$$

we have

$$\delta_s \dot{q}_n = \epsilon \dot{h}_n.$$

The first-order change in L is

$$\delta_s L = \sum_n \left(\frac{\partial L}{\partial q_n} \delta_s q_n + \frac{\partial L}{\partial \dot{q}_n} \delta_s \dot{q}_n \right).$$

Hence

$$\frac{\delta_s L}{\epsilon} = \sum_n \left(\frac{\partial L}{\partial q_n} h_n + p_n \dot{h}_n \right).$$

On a physical trajectory,

$$\frac{\partial L}{\partial q_n} = \dot{p}_n,$$

so

$$\frac{\delta_s L}{\epsilon} = \sum_n \left(\dot{p}_n h_n + p_n \dot{h}_n \right) = \frac{d}{dt} \left(\sum_n p_n h_n \right) = \dot{J}.$$

Thus pointwise Lagrangian invariance,

$$\delta_s L = 0,$$

gives

$$\dot{J} = 0.$$

This is the local form of the same result obtained globally from the action boundary term. The product rule is the mechanism.

13 A slightly more general symmetry: invariance up to a total derivative

Exact pointwise invariance of L is the simplest case, but the action can possess a symmetry even when the Lagrangian changes by a total time derivative. Suppose

$$\delta_s L = \epsilon \frac{dB}{dt}.$$

Then

$$\frac{d}{dt} \left(\sum_n p_n h_n \right) = \frac{dB}{dt}$$

along a physical trajectory, so

$$J_B \equiv \sum_n p_n h_n - B = \text{constant}.$$

Equivalently, the action changes only by an endpoint contribution. This extension emphasizes that the action, rather than pointwise invariance of L alone, is the fundamental object in the variational formulation.

14 Time translation and energy

Everything above transformed configuration variables while holding the time coordinate fixed:

$$\delta_s t = 0.$$

A time translation changes the independent variable itself, so we treat it separately rather than forcing it into the fixed-time derivation.

Define

$$E_L \equiv \sum_j p_j \dot{q}_j - L(\mathbf{q}, \dot{\mathbf{q}}, t),$$

with

$$p_j = \frac{\partial L}{\partial \dot{q}_j}.$$

Taking a total time derivative,

$$\begin{aligned} \dot{E}_L &= \sum_j (\dot{p}_j \dot{q}_j + p_j \ddot{q}_j) - \frac{dL}{dt} \\ &= \sum_j (\dot{p}_j \dot{q}_j + p_j \ddot{q}_j) - \left[\sum_j \frac{\partial L}{\partial q_j} \dot{q}_j + \sum_j \frac{\partial L}{\partial \dot{q}_j} \ddot{q}_j + \frac{\partial L}{\partial t} \right] \\ &= \sum_j (\dot{p}_j \dot{q}_j + p_j \ddot{q}_j) - \sum_j \dot{p}_j \dot{q}_j - \sum_j p_j \ddot{q}_j - \frac{\partial L}{\partial t}. \end{aligned}$$

Here we used

$$p_j = \frac{\partial L}{\partial \dot{q}_j}$$

by definition and

$$\dot{p}_j = \frac{\partial L}{\partial q_j}$$

from the Euler–Lagrange equations. The first two pairs of terms cancel, leaving

$$\dot{E}_L = -\frac{\partial L}{\partial t}.$$

Therefore, if

$$\frac{\partial L}{\partial t} = 0,$$

then

$$E_L = \text{constant}.$$

No explicit t means that the dynamical description has no preferred time origin.

14.1 When does E_L become $T + V$?

For an ordinary mechanical system

$$L = T - V,$$

with T quadratic in the generalized velocities and V independent of the velocities, Euler’s theorem for homogeneous quadratic functions gives

$$\sum_j \frac{\partial T}{\partial \dot{q}_j} \dot{q}_j = 2T.$$

Since

$$p_j = \frac{\partial L}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j},$$

we obtain

$$E_L = 2T - (T - V) = T + V.$$

Thus, in the familiar case,

$$E_L = T + V.$$

The more general conserved quantity is $E_L = \sum_j p_j \dot{q}_j - L$; it need not coincide with the naive expression $T + V$ for more general Lagrangians.

15 Physical reading and summary

The main physical examples can be summarized as

translation symmetry + dynamics	$\longrightarrow$	$\mathbf{P}$ conserved,
rotation symmetry + dynamics	$\longrightarrow$	$\mathbf{L}$ conserved,
time-translation symmetry + dynamics	$\longrightarrow$	E_L conserved.

The table is not the theorem. The theorem is the mechanism: a continuous symmetry of the action, together with the equations of motion, turns the action boundary term into a conserved quantity.

The first and second halves of this lecture are both statements about configuration-space geometry. For constraints, allowed virtual displacements are tangent while ideal reactions are normal. Solving the constraint keeps only the tangent degrees of freedom; retaining redundant coordinates and adding multipliers recovers the normal reactions. For symmetry, we identify special infinitesimal directions in configuration space that leave the action invariant, and the corresponding boundary quantity is conserved along physical motion.

A final caution is useful. A subsystem can exchange momentum or energy with degrees of freedom that have been omitted from the model. An externally driven model can contain explicit time dependence even when a larger closed system would not. Thus an apparent failure of a conservation law can reflect the choice of an open or reduced description rather than a failure of the underlying mechanics.

The central conceptual chains are

constraint surface $\rightarrow$ normal reaction direction $\rightarrow$ multiplier $\rightarrow L_\lambda$,
cyclic coordinate $\rightarrow$ continuous symmetry $\rightarrow$ action boundary term $\rightarrow$ Noether quantity.