

Graduate Classical Mechanics

Lecture 4: Discrete Dynamics and the Action Principle

From initial-value updates to stationary paths

1 Why change viewpoints?

Lecture 1 began with a very broad idea: a complete state contains enough information to determine what comes next,

$$S_{n+1} = F(S_n).$$

Lecture 2 showed how differences convert node values into edge values and how repeated differences expose structure. Lecture 3 returned to Newtonian mechanics and wrote a second-order equation as a first-order state-space evolution.

In this lecture we will put these ideas together and then change viewpoint.

We will first describe mechanics using only an *ordered sequence of configurations*. At that level there is not yet a physical clock. We will then add a clock and see how steps become velocities and turns become accelerations. Mass and force enter only after this kinematic structure is clear.

Then we will ask a very different question. Instead of constructing a trajectory one step at a time from initial data, suppose the initial and final configurations are fixed. Can the physical trajectory be characterized as a stationary point of one scalar quantity associated with the *whole path*?

That scalar quantity is the *action*. The main goal of this lecture is to see how a stationary discrete action produces a local difference equation at every interior point of the path. The continuum variational principle will come next.

Main ideas

By the end of the lecture we want to understand the chain

ordered sequence $\longrightarrow$ clock $\longrightarrow$ Newtonian dynamics $\longrightarrow$ stationary discrete action
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and the contrast

Newton: build the path from initial data	$\longleftrightarrow$	Hamilton: compare paths between fixed boundaries.
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2 The broad Newtonian law

The most useful general starting point for Newtonian mechanics is not necessarily $m\ddot{x} = F$, but

$\frac{d\mathbf{p}}{dt} = \mathbf{F}.$
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The momentum must then be related to the velocity. In the familiar nonrelativistic case,

$$\mathbf{p} = M\dot{\mathbf{x}}.$$

If the mass matrix M is constant, this gives

$$M\ddot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}, t).$$

The explicit dependence on $\dot{\mathbf{x}}$ or t can arise naturally when we describe only a subsystem. Friction, prescribed driving, moving supports, or a variable-mass subsystem can all lead to effective equations whose right-hand side depends on velocity or on an externally prescribed clock. The larger closed system contains additional degrees of freedom that have been left out of the reduced description.

We will later connect the absence of a preferred time origin to time-translation symmetry and energy conservation. For now our purpose is simpler: before discussing force, mass, or even seconds, let us isolate the geometry of an ordered sequence.

3 Mechanics before the clock

Consider an ordered sequence of configurations

$$\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \dots$$

in a vector space. The label n tells us only the order. It does not yet tell us how much physical time passes between two successive configurations.

Define the *step*

$$\mathbf{s}_n = \mathbf{x}_{n+1} - \mathbf{x}_n.$$

The step tells us how the configuration changes from one stage to the next. It is velocity-like, but it is not yet a velocity because there is no clock.

Now compare two successive steps. Define the *turn*

$$\boldsymbol{\kappa}_{n+1} = \mathbf{s}_{n+1} - \mathbf{s}_n.$$

The turn tells us how the step itself changes. It is acceleration-like, but it is not yet an acceleration. The word *turn* should not be read as meaning that the direction must rotate. In more than one dimension, $\boldsymbol{\kappa}_{n+1}$ can change the length of the step, its direction, or both. If $\boldsymbol{\kappa}_{n+1}$ points in the same direction as $\mathbf{s}_n$, the next step is lengthened; if it points in the opposite direction, the next step is shortened (and, if large enough, reversed). A transverse component changes the direction. Thus the cases usually called “speeding up,” “slowing down,” and “turning” are all contained in the single vector difference

$$\boldsymbol{\kappa}_{n+1} = \mathbf{s}_{n+1} - \mathbf{s}_n.$$

3.1 A geometric picture: move, then turn

Figure 1 shows three configurations in a two-dimensional configuration space. The solid vectors are the actual steps. The dashed copy of $\mathbf{s}_0$ begins at $\mathbf{x}_1$ and shows where the system would go if the next step were unchanged. The correction from that inertial prediction to the actual next point is the turn $\boldsymbol{\kappa}_1$.

The geometry says simply

$$\mathbf{s}_1 = \mathbf{s}_0 + \boldsymbol{\kappa}_1.$$

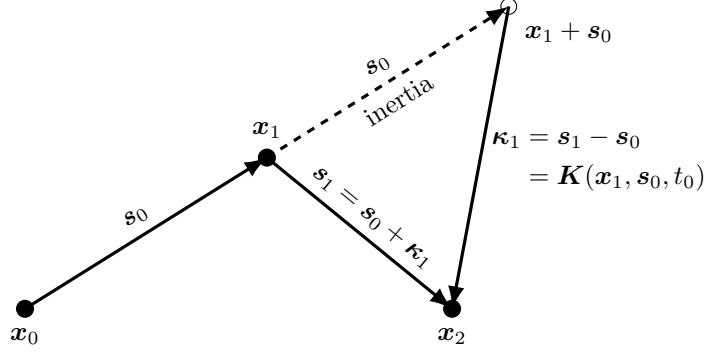


Figure 1: A discrete Newtonian geometry. The dashed continuation is what inertia alone would produce. The turn κ_1 changes the next step. Time has not yet been assigned to the sequence.

A dynamical law tells us how the turn is chosen. A convenient semi-explicit form is

$$\begin{aligned} \mathbf{x}_{n+1} &= \mathbf{x}_n + \mathbf{s}_n, \\ \mathbf{s}_{n+1} &= \mathbf{s}_n + \mathbf{K}(\mathbf{x}_{n+1}, \mathbf{s}_n, t_n). \end{aligned}$$

We first move using the old step. Then, at the new configuration, the law tells us how the next step changes. No inverse problem is required for this update.

For the simplest conservative position-only rule,

$$\mathbf{K} = \mathbf{K}(\mathbf{x}_{n+1}).$$

At this stage $\mathbf{K}$ is a rule for the *turn*. It is not yet a physical force.

The state can be taken to be

$$S_n = (\mathbf{x}_n, \mathbf{s}_n),$$

so the construction is another realization of the Lecture 1 form

$$S_{n+1} = F(S_n).$$

3.2 Accumulating steps and turns

The first- and second-difference descriptions can also be written as sums. Since

$$\kappa_j = \mathbf{s}_j - \mathbf{s}_{j-1},$$

the changes telescope:

$$\mathbf{s}_n = \mathbf{s}_0 + \sum_{j=1}^n \kappa_j.$$

The step at stage n is the initial step plus all of the turns accumulated up to that stage.

Position is obtained by accumulating the steps:

$$\mathbf{x}_n = \mathbf{x}_0 + \sum_{i=0}^{n-1} \mathbf{s}_i.$$

Substituting the previous result gives the discrete double-sum form

$$\mathbf{x}_n = \mathbf{x}_0 + n\mathbf{s}_0 + \sum_{i=0}^{n-1} \sum_{j=1}^i \boldsymbol{\kappa}_j.$$

Equivalently, after collecting how many later steps contain each turn,

$$\mathbf{x}_n = \mathbf{x}_0 + n\mathbf{s}_0 + \sum_{j=1}^{n-1} (n-j)\boldsymbol{\kappa}_j.$$

These formulas are the sequence versions of “integrate acceleration once to get velocity, and twice to get position.” Once a clock is introduced, they will correspond to

$$\mathbf{v}(t) = \mathbf{v}(t_0) + \int_{t_0}^t \mathbf{a}(\tau) d\tau,$$

and

$$\mathbf{x}(t) = \mathbf{x}(t_0) + \mathbf{v}(t_0)(t - t_0) + \int_{t_0}^t d\tau \int_{t_0}^{\tau} \mathbf{a}(u) du.$$

For constant turn, the weighted sum immediately produces the discrete quadratic that we examine next.

4 Three simple sequence laws

Before adding a clock, it is useful to see how much follows from the turn law alone. We use one spatial dimension in the next examples so that the algebra is transparent.

4.1 No turn: discrete Newton I

If

$$K = 0,$$

then

$$s_{n+1} = s_n = s_0.$$

Therefore

$$x_n = x_0 + ns_0.$$

This is the geometric content of Newton’s first law: if nothing turns the step, the sequence continues uniformly.

Notice what has *not* been used. We have not introduced seconds, velocity, acceleration, mass, or force. We have only used ordering and the statement that the step does not change.

4.2 Constant turn

Suppose

$$K = K_0$$

is constant. Then

$$s_n = s_0 + nK_0.$$

Summing the steps gives

$$x_n = x_0 + ns_0 + \frac{n(n-1)}{2}K_0.$$

A constant turn produces a quadratic sequence. This is the discrete analogue of the quadratic position law associated with constant acceleration, but we have not yet called K_0 an acceleration.

4.3 A spring-like turn

Now choose

$$K(x) = -cx,$$

where c is a dimensionless parameter at this stage. The semi-explicit update is

$$\begin{aligned} x_{n+1} &= x_n + s_n, \\ s_{n+1} &= s_n - cx_{n+1}. \end{aligned}$$

Therefore

$$\begin{pmatrix} x_{n+1} \\ s_{n+1} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 \\ -c & 1-c \end{pmatrix}}_{\Phi_c} \begin{pmatrix} x_n \\ s_n \end{pmatrix}.$$

The determinant is

$$\det \Phi_c = (1-c) + c = 1.$$

Thus the map preserves area in the (x, s) plane. We will not pursue the finer question of which discrete scalar function is conserved. For now the structural point is simply that the conservative spring-like map preserves phase-space area.

Eliminating s_n gives

$$x_{n+1} - (2-c)x_n + x_{n-1} = 0.$$

If $0 < c < 4$, the solutions are oscillatory. Writing

$$2 - c = 2 \cos \theta$$

gives

$$\cos \theta = 1 - \frac{c}{2},$$

and hence

$$x_n = A \cos(n\theta) + B \sin(n\theta).$$

The oscillator already exists as an ordered sequence. The clock will tell us how to interpret the index n physically.

5 Now add a clock

Assign a physical time to each configuration. For equal spacing, let

$$t_n = t_0 + n\Delta,$$

where Δ is the time between successive events.

The configuration-space step becomes a velocity-like quantity when we divide by the time step:

$$\mathbf{v}_n \approx \frac{\mathbf{s}_n}{\Delta}.$$

Likewise, the turn becomes acceleration-like:

$$\mathbf{a}_{n+1} \approx \frac{\boldsymbol{\kappa}_{n+1}}{\Delta^2}.$$

Thus the clock converts

$$\text{step} \longrightarrow \text{velocity}, \quad \text{turn} \longrightarrow \text{acceleration}.$$

Only after this kinematic structure is in place does Newton's second law introduce mass and force:

$$M\mathbf{a} = \mathbf{F}.$$

Therefore the turn law corresponding to a physical force is

$$\boldsymbol{\kappa} \approx \Delta^2 M^{-1} \mathbf{F}.$$

The turn is acceleration-like, not force-like. Mass belongs in the dynamics that relates the two.

5.1 Difference-operator language

We can express the same structure using the difference language of the earlier lectures. Let A_t denote a forward difference operator along the ordered time sequence. Schematically,

$$\mathbf{x} \xrightarrow{A_t} \mathbf{s} \xrightarrow{A_t} \boldsymbol{\kappa}.$$

Because the two differences live on staggered sequences, the two matrices need not literally have the same dimensions, but they have the same nearest-neighbor difference pattern.

If we define the clocked difference operator

$$D_t = \frac{1}{\Delta} A_t,$$

then

$$\mathbf{v} = D_t \mathbf{x}, \quad \mathbf{a} = D_t \mathbf{v}.$$

Newton's law can therefore be written schematically as

$$D_t M D_t \mathbf{x} = \mathbf{F},$$

or

$$M^{-1} D_t M D_t \mathbf{x} = M^{-1} \mathbf{F}.$$

For constant mass this reduces to the familiar acceleration-equals-force-per-mass statement.

5.2 Uniform gravity: why mass cancels

For uniform gravity,

$$\mathbf{F} = M\mathbf{g}.$$

Newton's law says

$$M\mathbf{a} = M\mathbf{g},$$

so

$$\boxed{\mathbf{a} = \mathbf{g}.}$$

Mass has not been omitted. It enters on both sides of the dynamical equation and cancels. At the sequence level,

$$\boxed{\kappa = g\Delta^2.}$$

This gives a physical interpretation to the constant-turn sequence.

5.3 The spring: where mass and the clock enter

For a one-dimensional Hooke-law force,

$$F = -kx,$$

Newton gives

$$ma = -kx,$$

so

$$\boxed{a = -\frac{k}{m}x.}$$

Consequently,

$$\boxed{K = -cx, \quad c = \frac{k}{m}\Delta^2.}$$

Since

$$\omega = \sqrt{\frac{k}{m}},$$

we can write

$$\boxed{c = (\omega\Delta)^2.}$$

The dimensionless number c that controlled the geometry of the discrete sequence contains two distinct pieces of information:

$$\underbrace{\frac{k}{m}}_{\text{physical system}} \times \underbrace{\Delta^2}_{\text{clock}}.$$

This is a useful example of why it is conceptually helpful to separate ordering from physical time.

6 MATLAB: three turn laws

The same update code produces three qualitatively different motions. The first version is deliberately written without mass or physical time. It works only with the sequence variables x_n and s_n .

```
N = 40;
x = zeros(1,N+1);
s = zeros(1,N+1);

x(1) = 0;
s(1) = 1;

c = 0.10;

for n = 1:N
    x(n+1) = x(n) + s(n);      % move

    % Choose one turn law:
    % K = 0;                  % free motion
    % K = -1;                 % constant turn
    K = -c*x(n+1);           % spring-like turn

    s(n+1) = s(n) + K;        % turn
end

figure
plot(0:N,x,'o-')
xlabel('n')
ylabel('x_n')

figure
plot(x,s,'o-')
xlabel('x_n')
ylabel('s_n')
axis equal
```

The first plot shows the sequence x_n . The second shows the same evolution in the (x, s) state plane.

To give the spring model physical units, introduce m , k , and Δ , and compute

```
dt = 0.10;
m = 2.0;
k = 8.0;

c = (k/m)*dt^2;
```

Then the same dimensionless update uses the physically meaningful combination

$$c = \frac{k}{m} \Delta^2.$$

For uniform gravity, the corresponding constant turn is

$$K = g \Delta^2,$$

because the factor of mass cancels between $F = mg$ and $F = ma$.

7 From stationary points to stationary paths

Before assigning a scalar quantity to an entire path, recall a simpler idea from ordinary mechanics. An equilibrium of a one-dimensional potential occurs at a stationary point,

$$\frac{dV}{dx} = 0.$$

A stationary point need not be a minimum. It may be a local minimum, a local maximum, or—when there are several variables—a saddle point. The word *stationary* is therefore more accurate than *minimum* for the principle we are about to develop.

There is another way to write the same one-dimensional condition. Give the point a small displacement δx . To first order,

$$\delta V = \frac{dV}{dx} \delta x.$$

If $\delta V = 0$ for every small allowed δx , then $dV/dx = 0$.

For many coordinates this notation becomes even more useful. Let

$$\mathbf{x} = (x_1, x_2, \dots, x_N)^T.$$

A small variation is a vector

$$\delta \mathbf{x} = (\delta x_1, \delta x_2, \dots, \delta x_N)^T,$$

and

$$\delta V = \left(\frac{\partial V}{\partial \mathbf{x}} \right)^T \delta \mathbf{x} = (\nabla V)^T \delta \mathbf{x}.$$

Stationarity under *every* variation requires

$$\nabla V = 0.$$

This single statement accounts simultaneously for variations in every coordinate. In a high-dimensional landscape there are many directions in which one may vary the point, so saddle points become a natural possibility rather than an exceptional one.

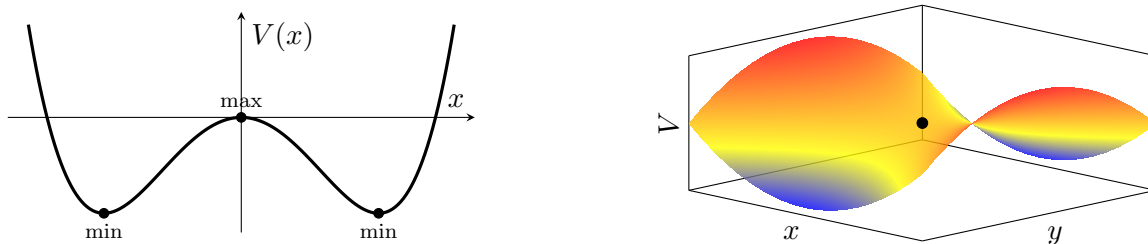


Figure 2: Stationary does not mean minimum. A one-variable function can have stationary maxima and minima. With several variables, a stationary point can also be a saddle, as in $V(x, y) = x^2 - y^2$.

7.1 From varying a point to varying a path

Now make one further generalization. Instead of a scalar function whose argument is a point, consider a scalar quantity whose argument is an *entire function* or path.

A simple geometric example is the distance between two fixed points in the plane. Suppose the path can be written as $x = x(y)$. Its arc length is

$$\ell[x] = \int_{y_1}^{y_2} ds = \int_{y_1}^{y_2} \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy.$$

We now vary the whole path while holding its endpoints fixed,

$$x(y) \longrightarrow x(y) + \delta x(y), \quad \delta x(y_1) = \delta x(y_2) = 0.$$

The straight line is characterized by

$$\delta \ell = 0.$$

For this particular problem the stationary path is a minimum, but stationarity is the more general idea.

Optics supplies a closely related example. If the local speed of light in a medium is $u(x, y)$, the travel time along a path is

$$T[x] = \int \frac{ds}{u(x, y)} = \int_{y_1}^{y_2} \frac{\sqrt{1 + (dx/dy)^2}}{u(x, y)} dy.$$

Equivalently, if $n(x, y)$ is the refractive index and c_0 is the vacuum speed of light,

$$T[x] = \frac{1}{c_0} \int_{y_1}^{y_2} n(x, y) \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy.$$

Fermat's principle states that the physical ray makes this travel time *stationary* under nearby fixed-endpoint variations. In familiar examples it is often a minimum, but the variational statement is stationarity.

Both examples have the general form

$$\mathcal{J}[x] = \int f(x, x', y) dy, \quad x' = \frac{dx}{dy},$$

and we seek paths for which

$$\delta \mathcal{J} = 0$$

for every allowed small variation of the interior of the path.

This is the functional analogue of $\delta V = 0$. For an ordinary function, stationarity under every coordinate variation gives algebraic equations such as $\nabla V = 0$. For a functional, stationarity under every local path variation will give a *differential equation at every point along the path*. The discrete derivation below will make that localization visible before we take any continuum limit.

7.2 Mechanics: choose time as the path parameter

For mechanics, replace the path parameter y by time t , and replace $x' = dx/dy$ by $\dot{x} = dx/dt$. Assume that there is a function

$$L(x, \dot{x})$$

and define the *action*

$$\mathcal{S}[x] = \int_{t_0}^{t_1} L(x, \dot{x}) dt.$$

For now we take L to have no explicit dependence on t . We also do *not* yet assume that $L = T - V$. We first ask what stationarity of a generic action implies.

The action is a *functional*: it assigns one scalar number to an entire candidate history $x(t)$. We hold the endpoints fixed and demand

$$\delta\mathcal{S} = 0$$

for every sufficiently small variation of the interior of the path. The next step is to make this statement discrete so that we can see, node by node, how one integral-like quantity produces a local equation of motion.

8 Space and time meet in the slope

To discretize the action we return to the space-time picture. Consider three successive events

$$P_{n-1} = (t_{n-1}, x_{n-1}), \quad P_n = (t_n, x_n), \quad P_{n+1} = (t_{n+1}, x_{n+1}),$$

with equal time spacing

$$t_{n-1} = (n-1)\Delta, \quad t_n = n\Delta, \quad t_{n+1} = (n+1)\Delta.$$

The vertical differences are the same configuration-space quantities introduced earlier:

$$s_{n-1} = x_n - x_{n-1}, \quad s_n = x_{n+1} - x_n,$$

and

$$\kappa_n = s_n - s_{n-1}.$$

They live in the x -direction. The time differences

$$t_n - t_{n-1} = \Delta, \quad t_{n+1} - t_n = \Delta$$

live in the horizontal direction.

A new variable appears only when space and time are compared. The discrete velocities are the slopes connecting neighboring events:

$$\dot{x}_{n-1} \approx \frac{x_n - x_{n-1}}{\Delta} = \frac{s_{n-1}}{\Delta},$$

$$\dot{x}_n \approx \frac{x_{n+1} - x_n}{\Delta} = \frac{s_n}{\Delta}.$$

Thus a point in the t - x plane is an *event*; a discrete $\dot{x}$ is associated with the connection from one event to the next.

where C is constant under this one-node variation and, with a slight abuse of notation,

$$\boxed{\mathcal{S}_d(x_n) = L\left(x_n, \frac{x_{n+1} - x_n}{\Delta}\right) \Delta + L\left(x_{n-1}, \frac{x_n - x_{n-1}}{\Delta}\right) \Delta.}$$

Three points are the minimum picture because the middle node belongs to both the incoming and outgoing connections.

The stationary-path condition must hold independently at every interior node:

$$\boxed{\frac{\partial \mathcal{S}_d}{\partial x_n} = 0, \quad n = 1, \dots, N-1.}$$

This is the key localization step. A single scalar statement about the whole sum produces a local difference equation at every interior point. In the continuum limit the same idea turns stationarity of an integral into a differential equation that must hold at every time.

10 Deriving the discrete Euler–Lagrange equation

We now make the dependence on the middle point x_n explicit. Define

$$\dot{x}_n(x_n) = \frac{x_{n+1} - x_n}{\Delta}, \quad \dot{x}_{n-1}(x_n) = \frac{x_n - x_{n-1}}{\Delta},$$

and

$$L_n = L(x_n, \dot{x}_n), \quad L_{n-1} = L(x_{n-1}, \dot{x}_{n-1}).$$

Then the part of the action that changes when x_n changes is

$$\boxed{\mathcal{S}_d(x_n) = \left[L_n(x_n, \dot{x}_n(x_n)) + L_{n-1}(x_{n-1}, \dot{x}_{n-1}(x_n)) \right] \Delta.}$$

Written this way, the chain rule is visible. The first term depends on x_n in two ways: explicitly through its first argument, and implicitly through the outgoing slope $\dot{x}_n$. The previous term has fixed position argument x_{n-1} , but it depends on x_n through the incoming slope $\dot{x}_{n-1}$.

Differentiate before simplifying:

$$\boxed{\frac{\partial \mathcal{S}_d(x_n)}{\partial x_n} = \left[\frac{\partial L_n}{\partial x_n} + \frac{\partial L_n}{\partial \dot{x}_n} \frac{\partial \dot{x}_n}{\partial x_n} + \frac{\partial L_{n-1}}{\partial \dot{x}_{n-1}} \frac{\partial \dot{x}_{n-1}}{\partial x_n} \right] \Delta.}$$

The two slope derivatives have opposite signs:

$$\frac{\partial \dot{x}_n}{\partial x_n} = \frac{\partial}{\partial x_n} \left(\frac{x_{n+1} - x_n}{\Delta} \right) = -\frac{1}{\Delta},$$

while

$$\frac{\partial \dot{x}_{n-1}}{\partial x_n} = \frac{\partial}{\partial x_n} \left(\frac{x_n - x_{n-1}}{\Delta} \right) = +\frac{1}{\Delta}.$$

Therefore

$$\frac{\partial \mathcal{S}_d(x_n)}{\partial x_n} = \Delta \frac{\partial L_n}{\partial x_n} - \frac{\partial L_n}{\partial \dot{x}_n} + \frac{\partial L_{n-1}}{\partial \dot{x}_{n-1}}.$$

Now vary the middle node by an arbitrary amount δx_n . To first order,

$$\delta \mathcal{S}_d = \frac{\partial \mathcal{S}_d}{\partial x_n} \delta x_n.$$

Using the previous result,

$$\delta \mathcal{S}_d = - \left[\frac{\frac{\partial L_n}{\partial \dot{x}_n} - \frac{\partial L_{n-1}}{\partial \dot{x}_{n-1}}}{\Delta} - \frac{\partial L_n}{\partial x_n} \right] \Delta \delta x_n.$$

Stationarity requires $\delta \mathcal{S}_d = 0$. The spacing Δ is not zero, and the interior variation δx_n is arbitrary. Therefore the quantity in brackets must vanish:

$$\frac{\frac{\partial L_n}{\partial \dot{x}_n} - \frac{\partial L_{n-1}}{\partial \dot{x}_{n-1}}}{\Delta} - \frac{\partial L_n}{\partial x_n} = 0, \quad n = 1, \dots, N-1.$$

This is the discrete Euler–Lagrange equation for the particular left-endpoint discretization we chose.

The first term is a discrete time derivative of $\partial L / \partial \dot{x}$. In the continuum limit,

$$\frac{\frac{\partial L_n}{\partial \dot{x}_n} - \frac{\partial L_{n-1}}{\partial \dot{x}_{n-1}}}{\Delta} \longrightarrow \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right).$$

Thus stationarity at every interior node becomes

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0.$$

We will derive this continuum equation carefully in the next lecture. Here the important point is that we can already see *why* its two terms appear:

$$\begin{array}{ll} \text{explicit dependence on the node } x_n & \longrightarrow \partial L / \partial x, \\ \text{dependence through the two adjacent slopes} & \longrightarrow \text{a time difference of } \partial L / \partial \dot{x}. \end{array}$$

Because this condition must hold for every interior n , one stationary global sum produces a local equation of motion all along the path.

11 Generalized momentum emerges

The discrete Euler–Lagrange equation suggests defining

$$p_n = \frac{\partial L_n}{\partial \dot{x}_n}.$$

This quantity is called the *generalized momentum*. The discrete equation becomes

$$\frac{p_n - p_{n-1}}{\Delta} = \frac{\partial L_n}{\partial x_n}.$$

This already resembles Newton's broad law

$$\dot{p} = F.$$

The resemblance is not accidental. We can now ask what choices of L make the generalized momentum and the right-hand side become the familiar mechanical quantities.

12 Example 1: the free particle

Take

$$L = \frac{1}{2}m\dot{x}^2.$$

Then

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x},$$

so the generalized momentum is the ordinary Newtonian momentum.

Because this L has no dependence on x ,

$$\frac{\partial L}{\partial x} = 0.$$

The discrete Euler–Lagrange equation gives

$$\frac{p_n - p_{n-1}}{\Delta} = 0,$$

so

$$p_n = p_{n-1}.$$

For constant mass,

$$\dot{x}_n = \dot{x}_{n-1}.$$

Equivalently,

$$x_{n+1} - 2x_n + x_{n-1} = 0.$$

Thus the same free-particle law that appeared earlier as $K = 0$ now appears from stationarity of the discrete action.

This is our first indication that the initial-value and variational descriptions may be two organizations of the same dynamics.

13 Example 2: discovering the mechanical Lagrangian

There is a useful way to see where the familiar combination $T - V$ comes from before we use it in the action. Start with Newton's equation for a simple conservative one-dimensional system,

$$\boxed{\frac{d}{dt}(m\dot{x}) = F(x) = -\frac{dV}{dx}.$$

Now define the kinetic energy

$$T(\dot{x}) = \frac{1}{2}m\dot{x}^2.$$

Then the momentum can itself be recognized as a derivative:

$$\boxed{m\dot{x} = \frac{\partial T}{\partial \dot{x}}.$$

For this simple Cartesian system, T depends only on $\dot{x}$ and V depends only on x . Therefore the “crossed” derivatives vanish:

$$\frac{\partial T}{\partial x} = 0, \quad \frac{\partial V}{\partial \dot{x}} = 0.$$

Newton's equation can therefore be rewritten as

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}} \right) + \frac{\partial V}{\partial x} = 0.$$

Using the two zero derivatives above, this becomes

$$\frac{d}{dt} \left(\frac{\partial(T - V)}{\partial \dot{x}} \right) - \frac{\partial(T - V)}{\partial x} = 0,$$

or, compactly,

$$\boxed{\left(\frac{d}{dt} \frac{\partial}{\partial \dot{x}} - \frac{\partial}{\partial x} \right) (T - V) = 0.}$$

This is exactly the Euler–Lagrange operator acting on $T - V$. In this sense we can read the logic in reverse: for ordinary conservative Cartesian mechanics, Newton's law already contains the combination $T - V$. The action principle will tell us why this differential structure arises from stationarity of a whole path.

We can see the same result by asking directly what the discrete Euler–Lagrange equation requires. Suppose we want

$$\dot{p} = F(x), \quad F(x) = -V'(x).$$

The discrete Euler–Lagrange equation says

$$\frac{p_n - p_{n-1}}{\Delta} = \frac{\partial L_n}{\partial x_n}.$$

If we want

$$p = m\dot{x},$$

then the velocity-dependent part of L must satisfy

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x},$$

which gives

$$T(\dot{x}) = \frac{1}{2}m\dot{x}^2.$$

If we also want

$$\frac{\partial L}{\partial x} = F(x) = -V'(x),$$

then the position-dependent part must be $-V(x)$. Thus

$$L(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 - V(x) = T - V.$$

The minus sign is not arbitrary: it is what makes

$$\frac{\partial L}{\partial x} = -V'(x) = F(x).$$

This argument applies to the simple conservative form considered here. More general Lagrangians can contain explicit time dependence, velocity-dependent interactions, fields, and other structures.

Recall the result from Lecture 3: for a conservative system the energy is

$$E = T + V.$$

So two closely related combinations appear for different structural reasons:

$$E = T + V \quad \text{while} \quad L = T - V.$$

The sum is associated with conserved mechanical energy; the difference is the quantity whose stationary action generates the motion.

14 Newton's law returns

For

$$L = \frac{1}{2}m\dot{x}^2 - V(x),$$

we have

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}, \quad \frac{\partial L}{\partial x} = -V'(x).$$

The continuum-looking form of Euler–Lagrange is therefore

$$\frac{d}{dt}(m\dot{x}) = -V'(x),$$

or

$$m\ddot{x} = F(x).$$

At the discrete level,

$$\frac{m}{\Delta} \left(\frac{x_{n+1} - x_n}{\Delta} - \frac{x_n - x_{n-1}}{\Delta} \right) = -V'(x_n),$$

so

$$m \frac{x_{n+1} - 2x_n + x_{n-1}}{\Delta^2} = -V'(x_n).$$

Now define

$$s_{n-1} = x_n - x_{n-1}, \quad s_n = x_{n+1} - x_n.$$

Then the same equation becomes

$$s_n = s_{n-1} - \frac{\Delta^2}{m} V'(x_n).$$

Together with

$$x_n = x_{n-1} + s_{n-1},$$

we recover exactly the move-then-turn structure from the beginning of the lecture:

$$\begin{array}{ll} \text{move:} & x_n = x_{n-1} + s_{n-1}, \\ \text{turn:} & s_n = s_{n-1} - \frac{\Delta^2}{m} V'(x_n), \end{array}$$

with

$$K(x_n) = -\frac{\Delta^2}{m} V'(x_n).$$

This closes the circle. We began by describing Newtonian mechanics as a local state update. We then asked for a scalar quantity associated with a whole path whose stationary states reproduce the same motion. The discrete action does exactly that.

15 Initial-value and boundary-value viewpoints

It is worth making the contrast explicit.

Newtonian update viewpoint	Variational viewpoint
Specify an initial state (x_0, s_0) .	Fix the boundary configurations x_0, x_N .
Use the local law to construct the next state.	Compare complete candidate paths between the boundaries.
Repeat: $S_0 \rightarrow S_1 \rightarrow S_2 \rightarrow \dots$.	Require the action to be stationary under every interior variation.
The path is built locally.	A global-looking scalar condition produces a local equation at every node.

These are not different physical theories. In the conservative mechanical examples above, they produce the same discrete equation of motion. They organize the information differently.

16 What comes next

We have worked entirely with a discrete path because the structure is visible there. At an interior node x_n , only two adjacent pieces of the action respond to the variation. Requiring stationarity at every node gives a local difference equation.

The next step is to let the time spacing become small. The discrete sum

$$\mathcal{S}_d = \sum_n L(x_n, \dot{x}_n) \Delta$$

becomes

$$\boxed{\mathcal{S}[x] = \int L(x, \dot{x}, t) dt.}$$

The neighboring finite differences become derivatives, and the discrete difference of generalized momentum becomes

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}}.$$

In the continuum derivation, the role played here by the two neighboring edges will appear as an integration-by-parts boundary term. This will lead to Hamilton's principle and the Euler–Lagrange equations in their general form.

Structural summary. Ordering gives an adjacent configuration. A clock converts configuration differences into rates. Newton relates acceleration to force through mass. The action principle changes viewpoint: fix the boundaries, vary the interior path, and demand stationarity. Because each interior node enters only its neighboring action pieces, stationarity of one global sum becomes a local equation at every node.