Lecture 18 Instructor Outline — Draft

Generating Functions and Canonical Transformations in Practice

PHYS V1100: Analytical Dynamics — 105 minutes

How do we construct useful canonical coordinates?

Core arc:

$$Q = Q(q,p),\ P = P(q,p), \qquad \alpha = p\,dq - P\,dQ = A\,dq + B\,dp$$

$$\Downarrow \text{ Green/Stokes} + \{Q,P\} = 1$$

$$\oint_{\partial\Sigma} \alpha = 0 \quad \Longrightarrow \quad \text{path independence}$$

$$\Downarrow$$

$$\boldsymbol{p}^T d\boldsymbol{q} - \boldsymbol{P}^T d\boldsymbol{Q} = dF$$

$$\Downarrow \text{ restore time}$$

$$\boldsymbol{p}^T d\boldsymbol{q} - H\,dt = \boldsymbol{P}^T d\boldsymbol{Q} - K\,dt + dF$$

$$\Downarrow \text{ harmonic oscillator: see the destination before the machinery}$$

$$L = \tfrac{1}{2}(\dot{q}^2 - q^2) \xrightarrow{\text{Legendre}} H = \tfrac{1}{2}(p^2+q^2) \xrightarrow{F_1(q,Q)} K = P \xrightarrow{G_2(Q,\bar{P},t)} \bar{K} = 0$$

$$\Downarrow \text{ now organize the available representations}$$

$$F_1(\boldsymbol{q},\boldsymbol{Q},t) \longleftrightarrow F_2(\boldsymbol{q},\boldsymbol{P},t) \longleftrightarrow F_3(\boldsymbol{p},\boldsymbol{Q},t) \longleftrightarrow F_4(\boldsymbol{p},\boldsymbol{P},t)$$

$$\Downarrow$$

Hamilton–Jacobi: can we construct $K = 0$ directly for a general H?

Protected goals.

- Make the existence of a generating function explicit in ordinary multivariable calculus before using exterior-form compression: in one (q,p) plane, calculate the circulation of

  $$\alpha = p\,dq - P\,dQ$$

  and show that canonicality makes it vanish.

- Use a two-path figure and Green's theorem to turn zero circulation into path independence, then define

  $$F(z_b) - F(z_a) = \int_{z_a}^{z_b} \alpha$$

  so that $\alpha = dF$ is a conclusion rather than an assumption.

- Generalize concretely to $\boldsymbol{q} = (q_1, q_2)$, $\boldsymbol{p} = (p_1, p_2)$ by restricting first to the (q_1, p_1) plane, then expose the pairwise sums that extend to arbitrary (q_j, p_k), (q_j, q_k), and (p_j, p_k) planes.

- State the exterior-form summary $d\alpha = 0$ only after the coordinate calculation, as compact notation for a result students can already compute.

- Restore time only after the fixed-time geometry is clear, obtaining

  $$\boldsymbol{p}^T d\boldsymbol{q} - H\,dt = \boldsymbol{P}^T d\boldsymbol{Q} - K\,dt + dF,$$

  so the phase-space action changes only by endpoint data.

- Use the unit-frequency harmonic oscillator as the central construction example. Begin from

  $$L = \frac{1}{2}(\dot{q}^2 - q^2)$$

  and review the Legendre transform quickly to obtain $H = \frac{1}{2}(p^2+q^2)$ before using generating functions heavily.

- Choose canonical variables adapted to the oscillator circles: $P = H$ labels the orbit while a local phase Q advances uniformly. Construct this transformation with $F_1(q,Q)$, giving $K = P$.

- Then use an explicitly time-dependent type-2 generating function to remove the remaining uniform phase advance and obtain $\bar{K} = 0$. Use this concrete sequence to motivate why different choices of independent variables for a generating function are useful.

- Only after the oscillator example, organize F_1, F_2, F_3, F_4 structurally as partial Legendre transforms. Do not teach the four types as a sign table to memorize.

- Keep a short identity/scaling check and recover the Lecture 16 infinitesimal-generator law from $F_2 = \boldsymbol{q}^T\boldsymbol{P} + \epsilon G$.
- End by asking whether a general Hamiltonian admits a generating function that reaches $K = 0$ directly. Display the Hamilton–Jacobi equation as the Lecture 19 handoff; keep action–angle variables for Lecture 20.

Notation and scope. Vectors are columns. The opening circulation argument uses explicit component derivatives because the point is to make the calculation executable with ordinary multivariable calculus. Assume the coordinate transformation is smooth enough that mixed second derivatives commute. For scalar generating functions, $\partial F/\partial \boldsymbol{q}$, $\partial F/\partial \boldsymbol{p}$, $\partial F/\partial \boldsymbol{Q}$, and $\partial F/\partial \boldsymbol{P}$ are column gradients. All generating-function statements are local: path independence and the existence of F are asserted on a sufficiently small simply connected neighborhood, and each choice of independent variables requires the corresponding local invertibility.

Pacing. The first 27 minutes make the generating function visible by a two-dimensional circulation calculation and then generalize the result plane by plane to $2N$-dimensional phase space. Minutes 27–36 restore time and connect the exact differential to the phase-space action. Minutes 36–64 use the harmonic oscillator to review $L \to H$, construct $K = P$ with an F_1 representation, and then use an explicitly time-dependent type-2 function to reach $\bar{K} = 0$. Minutes 64–87 organize the four generating-function representations and use quick checks. The final 18 minutes reconnect to infinitesimal generators and turn the oscillator's $\bar{K} = 0$ endpoint into the Hamilton–Jacobi question.

0–17 min: Why a generating function exists: calculate the two-dimensional case

BOARD Work first in an actual two-dimensional phase plane,

$$Q = Q(q,p), \qquad P = P(q,p),$$

and compare the old and new canonical line elements:

$$\boxed{\alpha \equiv p\,dq - P\,dQ.}$$

Since

$$dQ = \frac{\partial Q}{\partial q}\,dq + \frac{\partial Q}{\partial p}\,dp,$$

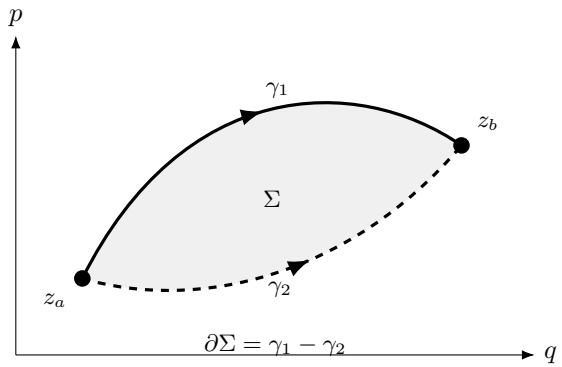
write

$$\boxed{\alpha = A(q,p)\,dq + B(q,p)\,dp}$$

with

$$A = p - P\frac{\partial Q}{\partial q}, \qquad B = -P\frac{\partial Q}{\partial p}.$$

BOARD / PROJECT Two different paths from the same point z_a to the same point z_b:



SAY Traverse γ_1 from z_a to z_b and return along γ_2. Their difference is the closed boundary $\partial\Sigma = \gamma_1 - \gamma_2$. Green's theorem gives

$$\int_{\gamma_1} \alpha - \int_{\gamma_2} \alpha = \oint_{\partial\Sigma} \alpha = \iint_{\Sigma} \left(\frac{\partial B}{\partial q} - \frac{\partial A}{\partial p}\right) dq\,dp.$$

BOARD Calculate the two derivatives rather than invoking exterior-form notation:

$$\frac{\partial B}{\partial q} = -\frac{\partial P}{\partial q}\frac{\partial Q}{\partial p} - P\frac{\partial^2 Q}{\partial q\,\partial p},$$

$$\frac{\partial A}{\partial p} = 1 - \frac{\partial P}{\partial p}\frac{\partial Q}{\partial q} - P\frac{\partial^2 Q}{\partial p\,\partial q}.$$

The mixed second derivatives cancel, so

$$\begin{aligned}\frac{\partial B}{\partial q} - \frac{\partial A}{\partial p} &= -1 + \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} \\ &= \boxed{\{Q,P\}_{q,p} - 1}.\end{aligned}$$

For a canonical transformation, $\{Q,P\}_{q,p} = 1$. Therefore

$$\boxed{\oint_{\partial\Sigma} \alpha = 0} \qquad \Longrightarrow \qquad \boxed{\int_{\gamma_1} \alpha = \int_{\gamma_2} \alpha.}$$

SAY The line integral depends only on the endpoints. Define the endpoint function by

$$\boxed{F(z_b) - F(z_a) = \int_{z_a}^{z_b} \alpha.}$$

Then, locally,

$$\boxed{dF = p\,dq - P\,dQ.}$$

The generating function has appeared because canonicality made the circulation vanish.

17–27 min: From one plane to four dimensions, then to $2N$

BOARD Now take

$$\boldsymbol{q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \qquad \boldsymbol{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix},$$

with Q_i, P_i allowed to depend on all four old phase-space coordinates. Examine only the (q_1, p_1) plane by holding q_2, p_2 fixed. Then

$$dQ_i = \frac{\partial Q_i}{\partial q_1}\,dq_1 + \frac{\partial Q_i}{\partial p_1}\,dp_1,$$

and

$$\alpha = \boldsymbol{p}^T d\boldsymbol{q} - \boldsymbol{P}^T d\boldsymbol{Q} = A_1\,dq_1 + B_1\,dp_1,$$

where

$$A_1 = p_1 - \sum_{i=1}^{2} P_i \frac{\partial Q_i}{\partial q_1}, \qquad B_1 = -\sum_{i=1}^{2} P_i \frac{\partial Q_i}{\partial p_1}.$$

The same mixed-derivative cancellation gives

$$\boxed{\frac{\partial B_1}{\partial q_1} - \frac{\partial A_1}{\partial p_1} = -1 + \sum_{i=1}^{2}\left(\frac{\partial Q_i}{\partial q_1}\frac{\partial P_i}{\partial p_1} - \frac{\partial Q_i}{\partial p_1}\frac{\partial P_i}{\partial q_1}\right).}$$

SAY The transformed terms have grouped themselves into the conjugate pairs (Q_i, P_i). The Lecture 17 condition

$$\mathsf{C}\mathsf{J}\mathsf{C}^T = \mathsf{J}$$

is equivalent, for an invertible canonical map, to

$$\boxed{\mathsf{C}^T\mathsf{J}\mathsf{C} = \mathsf{J}.}$$

Its (q_1, p_1) entry says the sum above is exactly 1, so this plane also has zero circulation.

BOARD Now replace (q_1, p_1) by arbitrary coordinate planes. The mixed plane (q_j, p_k) gives

$$\boxed{\sum_{i=1}^{N}\left(\frac{\partial Q_i}{\partial q_j}\frac{\partial P_i}{\partial p_k}-\frac{\partial Q_i}{\partial p_k}\frac{\partial P_i}{\partial q_j}\right)=\delta_{jk}.}$$

The (q_j, q_k) and (p_j, p_k) planes give, respectively,

$$\boxed{\sum_{i=1}^{N}\left(\frac{\partial Q_i}{\partial q_j}\frac{\partial P_i}{\partial q_k}-\frac{\partial Q_i}{\partial q_k}\frac{\partial P_i}{\partial q_j}\right)=0,}$$

$$\boxed{\sum_{i=1}^{N}\left(\frac{\partial Q_i}{\partial p_j}\frac{\partial P_i}{\partial p_k}-\frac{\partial Q_i}{\partial p_k}\frac{\partial P_i}{\partial p_j}\right)=0.}$$

SAY Every coordinate-plane circulation vanishes. Therefore the full phase-space one-form is locally path independent:

$$\boxed{\boldsymbol{p}^T d\boldsymbol{q}-\boldsymbol{P}^T d\boldsymbol{Q}=dF.}$$

OPTIONAL In exterior-form language, everything just calculated plane by plane is compressed into

$$d\alpha=0 \quad\Longrightarrow\quad \alpha=dF \ \text{ locally.}$$

Use this only as shorthand for the coordinate calculation, not as the primary derivation.

27–36 min: Restore time: the exact differential becomes an endpoint term

SAY Now that F is visible at fixed time, restore explicit time dependence. Along an actual trajectory,

$$\frac{dF}{dt}=\boldsymbol{p}^T\dot{\boldsymbol{q}}-\boldsymbol{P}^T\dot{\boldsymbol{Q}}+K-H,$$

which is equivalently

$$\boxed{\boldsymbol{p}^T d\boldsymbol{q}-H\,dt=\boldsymbol{P}^T d\boldsymbol{Q}-K\,dt+dF.}$$

Therefore the two phase-space action integrands differ only by a total derivative:

$$\boldsymbol{p}^T\dot{\boldsymbol{q}}-H=\boldsymbol{P}^T\dot{\boldsymbol{Q}}-K+\frac{dF}{dt}.$$

Integrating gives

$$\boxed{S=S'+F\big|_{t_i}^{t_f}.}$$

SAY This is why the generating function may change the endpoint bookkeeping without changing the interior equations of motion.

36–48 min: Oscillator target: review the Legendre transform and adapt coordinates to the circles

BOARD Start from a familiar Lagrangian with unit mass and unit frequency,

$$\boxed{L(q,\dot{q})=\frac{1}{2}\dot{q}^{\,2}-\frac{1}{2}q^2.}$$

One more quick Legendre-transform review:

$$p=\frac{\partial L}{\partial\dot{q}}=\dot{q},\qquad \dot{q}=p,$$

so

$$\begin{aligned} H(q,p) &= p\dot{q} - L \\ &= p^2 - \frac{1}{2}p^2 + \frac{1}{2}q^2 \\ &= \boxed{\frac{1}{2}(p^2+q^2)}. \end{aligned}$$

SAY This is the same Legendre-transform logic we will use again when we change which conjugate variables are independent in the generating function.

BOARD The phase-space trajectories satisfy

$$p^2 + q^2 = 2E.$$

Draw one circle in the (q,p) plane. Ask for coordinates adapted to this geometry. On a local angular branch, choose

$$\boxed{P = \frac{1}{2}(p^2+q^2), \qquad Q = \tan^{-1}\left(\frac{q}{p}\right).}$$

Here P is the conserved orbit/energy label; the oscillation amplitude is $\sqrt{2P}$. The angle Q is a local phase coordinate. Do not name these as action–angle variables yet; here they are simply coordinates adapted to this oscillator.

BOARD Verify canonicality directly:

$$\frac{\partial Q}{\partial q} = \frac{p}{p^2+q^2}, \qquad \frac{\partial Q}{\partial p} = -\frac{q}{p^2+q^2},$$

$$\frac{\partial P}{\partial q} = q, \qquad \frac{\partial P}{\partial p} = p,$$

so

$$\boxed{\{Q,P\} = \frac{p^2+q^2}{p^2+q^2} = 1.}$$

Therefore this is a canonical transformation and

$$\boxed{K(Q,P) = P.}$$

Hamilton's equations become

$$\boxed{\dot{P} = 0, \qquad \dot{Q} = 1,}$$

so

$$P = P_0, \qquad Q = Q_0 + t.$$

SAY We have turned oscillation of both q and p into one constant orbit label plus one phase that advances uniformly.

48–64 min: Construct the oscillator simplification: first F_1, then a time-dependent type-2 function

BOARD The transformation itself tells us which independent variables are convenient. From

$$q = \sqrt{2P}\sin Q, \qquad p = \sqrt{2P}\cos Q,$$

we have locally

$$p = q\cot Q.$$

At fixed time, use

$$dF_1 = p\,dq - P\,dQ.$$

Since q and Q are the independent variables,

$$p = \frac{\partial F_1}{\partial q}.$$

Integrating with respect to q,

$$\boxed{F_1(q,Q) = \frac{1}{2}q^2 \cot Q}$$

(up to an irrelevant additive constant on this local branch). Then

$$\boxed{-\frac{\partial F_1}{\partial Q} = \frac{1}{2}q^2 \csc^2 Q = \frac{1}{2}(q^2+p^2) = P.}$$

Because F_1 has no explicit time dependence,

$$\boxed{K = H = P.}$$

SAY The first generating function did not merely prove that the map was canonical; it constructed coordinates in which the Hamiltonian itself became one of the new momenta.

BOARD But one piece of motion remains:

$$Q = Q_0 + t.$$

Move with that uniform phase. Define final variables

$$\boxed{\bar{Q} = Q - t, \qquad \bar{P} = P.}$$

The convenient independent variables are now Q and $\bar{P}$. This is the same conjugate-variable swap as a Legendre transform: replacing $\bar{Q}$ by $\bar{P}$ gives a type-2 differential of the form

$$dG_2 = P\,dQ + \bar{Q}\,d\bar{P} + (\bar{K} - K)dt.$$

Choose

$$\boxed{G_2(Q,\bar{P},t) = (Q-t)\bar{P}.}$$

Then

$$P = \frac{\partial G_2}{\partial Q} = \bar{P}, \qquad \bar{Q} = \frac{\partial G_2}{\partial \bar{P}} = Q - t,$$

and

$$\frac{\partial G_2}{\partial t} = -\bar{P}.$$

Therefore

$$\boxed{\bar{K} = K + \frac{\partial G_2}{\partial t} = \bar{P} - \bar{P} = 0.}$$

Hence

$$\boxed{\dot{\bar{Q}} = 0, \qquad \dot{\bar{P}} = 0.}$$

SAY The oscillator has shown the destination before the general machinery:

$$\boxed{L(q,\dot{q}) \xrightarrow{\text{Legendre}} H(q,p) \xrightarrow{F_1(q,Q)} K(Q,P) = P \xrightarrow{G_2(Q,\bar{P},t)} \bar{K} = 0.}$$

We used two different generating-function representations because two different pairs of independent variables were natural at different stages.

64–79 min: Organize the four generating-function representations after seeing why they are needed

BOARD Return to the general time-dependent relation

$$\boxed{\boldsymbol{p}^T d\boldsymbol{q} - H\,dt = \boldsymbol{P}^T d\boldsymbol{Q} - K\,dt + dF.}$$

If the natural independent variables are $(\boldsymbol{q}, \boldsymbol{Q}, t)$, write

$$F_1 = F_1(\boldsymbol{q}, \boldsymbol{Q}, t).$$

Coefficient matching gives

$$\boxed{\boldsymbol{p} = \frac{\partial F_1}{\partial \boldsymbol{q}}, \qquad \boldsymbol{P} = -\frac{\partial F_1}{\partial \boldsymbol{Q}}, \qquad K = H + \frac{\partial F_1}{\partial t}.}$$

BOARD To replace $\boldsymbol{Q}$ by its conjugate $\boldsymbol{P}$, make a partial Legendre transform:

$$\boxed{F_2 = F_1 + \boldsymbol{Q}^T \boldsymbol{P}.}$$

Then

$$dF_2 = \boldsymbol{p}^T d\boldsymbol{q} + \boldsymbol{Q}^T d\boldsymbol{P} + (K - H)dt,$$

so

$$\boxed{\boldsymbol{p} = \frac{\partial F_2}{\partial \boldsymbol{q}}, \qquad \boldsymbol{Q} = \frac{\partial F_2}{\partial \boldsymbol{P}}, \qquad K = H + \frac{\partial F_2}{\partial t}.}$$

SAY This is exactly the Legendre-transform idea reviewed at the start of the oscillator example: change which conjugate variable is treated as independent while preserving the differential structure.

BOARD The remaining two choices follow the same way:

$$\boxed{\begin{aligned} F_3 &= F_1 - \boldsymbol{q}^T \boldsymbol{p}, \\ F_4 &= F_1 + \boldsymbol{Q}^T \boldsymbol{P} - \boldsymbol{q}^T \boldsymbol{p}. \end{aligned}}$$

Summarize, rather than memorize,

function	variables	differential coefficients
F_1	$(\boldsymbol{q}, \boldsymbol{Q}, t)$	$\boldsymbol{p} = \partial F_1/\partial \boldsymbol{q}, \quad \boldsymbol{P} = -\partial F_1/\partial \boldsymbol{Q}$
F_2	$(\boldsymbol{q}, \boldsymbol{P}, t)$	$\boldsymbol{p} = \partial F_2/\partial \boldsymbol{q}, \quad \boldsymbol{Q} = \partial F_2/\partial \boldsymbol{P}$
F_3	$(\boldsymbol{p}, \boldsymbol{Q}, t)$	$\boldsymbol{q} = -\partial F_3/\partial \boldsymbol{p}, \quad \boldsymbol{P} = -\partial F_3/\partial \boldsymbol{Q}$
F_4	$(\boldsymbol{p}, \boldsymbol{P}, t)$	$\boldsymbol{q} = -\partial F_4/\partial \boldsymbol{p}, \quad \boldsymbol{Q} = \partial F_4/\partial \boldsymbol{P}$

For every representation,

$$\boxed{K = H + \frac{\partial F_i}{\partial t}.}$$

CHECK The oscillator used $F_1(q, Q)$ because (q, Q) were convenient, then $G_2(Q, \bar{P}, t)$ because $(Q, \bar{P})$ were convenient. The four types are local representations of one construction principle, not four theories.

79–87 min: Quick checks: identity and reciprocal scaling

BOARD Identity from the type-2 workhorse,

$$\boxed{F_2(\boldsymbol{q}, \boldsymbol{P}) = \boldsymbol{q}^T \boldsymbol{P}} \quad \Longrightarrow \quad \boxed{\boldsymbol{Q} = \boldsymbol{q}, \ \ \boldsymbol{P} = \boldsymbol{p}.}$$

BOARD Reciprocal scaling in one degree of freedom,

$$\boxed{F_2(q, P) = aqP}, \qquad a \neq 0,$$

gives

$$p = aP, \qquad Q = aq,$$

so

$$\boxed{Q = aq, \qquad P = \frac{p}{a}.}$$

SAY Lecture 17 checked these maps by the symplectic condition. Here one scalar constructs them directly.

87–95 min: Infinitesimal limit: reconnect generating functions to Poisson-bracket generators

BOARD Consider a near-identity transformation generated through

$$\boxed{F_2(\boldsymbol{q},\boldsymbol{P}) = \boldsymbol{q}^T\boldsymbol{P} + \epsilon G(\boldsymbol{q},\boldsymbol{P}), \qquad |\epsilon| \ll 1.}$$

Then

$$\boldsymbol{p} = \boldsymbol{P} + \epsilon\frac{\partial G}{\partial \boldsymbol{q}}, \qquad \boldsymbol{Q} = \boldsymbol{q} + \epsilon\frac{\partial G}{\partial \boldsymbol{P}}.$$

To first order, replace $\boldsymbol{P}$ by $\boldsymbol{p}$ inside the $O(\epsilon)$ terms:

$$\boxed{\delta\boldsymbol{q} = \epsilon\frac{\partial G}{\partial \boldsymbol{p}}, \qquad \delta\boldsymbol{p} = -\epsilon\frac{\partial G}{\partial \boldsymbol{q}}.}$$

Therefore, for any phase-space function A,

$$\boxed{\delta A = \epsilon\{A, G\}.}$$

SAY This is the Lecture 16 generator law recovered as the infinitesimal limit of the finite generating-function construction.

95–105 min: Hamilton–Jacobi handoff: can the oscillator's $K = 0$ trick be made general?

SAY The oscillator reached $\bar{K} = 0$ because we recognized its geometry and built two tailored canonical transformations. Now reverse the logic: for a general Hamiltonian, can we choose one generating function that makes the transformed Hamiltonian vanish directly?

BOARD Use a type-2 generating function

$$S(\boldsymbol{q},\boldsymbol{P},t).$$

If we demand

$$\boxed{K = 0,}$$

then

$$\boldsymbol{p} = \frac{\partial S}{\partial \boldsymbol{q}}, \qquad K = H\left(\boldsymbol{q}, \frac{\partial S}{\partial \boldsymbol{q}}, t\right) + \frac{\partial S}{\partial t}.$$

Therefore S must satisfy

$$\boxed{H\left(\boldsymbol{q}, \frac{\partial S}{\partial \boldsymbol{q}}, t\right) + \frac{\partial S}{\partial t} = 0.}$$

SAY If this succeeds, Hamilton's equations in the new variables say

$$\dot{\boldsymbol{P}} = \boldsymbol{0}, \qquad \dot{\boldsymbol{Q}} = \boldsymbol{0}.$$

The oscillator example showed concretely what that means: choose canonical coordinates in which the entire motion has been absorbed into the generating function.

NEXT Lecture 19 develops this Hamilton–Jacobi equation, complete solutions, separation, and the meaning of the constants carried by the new canonical variables.

STOP Do not solve a full Hamilton–Jacobi problem here, and do not introduce action–angle variables. Preserve those as the jobs of Lectures 19 and 20.