Lecture 17 Instructor Outline — Draft
Canonical Transformations and Symplectic Structure

PHYS V1100: Analytical Dynamics — 105 minutes

Which changes of phase-space coordinates preserve Hamiltonian mechanics?

Core arc:

$$(\boldsymbol{q},\boldsymbol{p}) \longrightarrow (\boldsymbol{Q},\boldsymbol{P})$$

$$\Downarrow \text{ what must survive the change of description?}$$

$$\{Q_i,Q_j\}=0, \quad \{P_i,P_j\}=0, \quad \{Q_i,P_j\}=\delta_{ij}$$

$$\Downarrow$$

$$N=1: \quad dQ \wedge dP = dq \wedge dp$$

$$\Downarrow \text{ generalize the conjugate pairing}$$

$$\omega = \textstyle\sum_{i=1}^{N} dq_i \wedge dp_i$$

$$\Downarrow$$

$$\mathsf{C}\mathsf{J}\mathsf{C}^T = \mathsf{J} \rightarrow \text{Poisson brackets and Hamilton's equations keep their form}$$

$$\Downarrow$$

$$\text{symplectic preservation} \Rightarrow \text{Liouville volume preservation}$$

$$\text{but for } N>1: \text{ volume preservation} \not\Rightarrow \text{symplectic preservation}$$

$$\Downarrow$$

$$\text{Hamiltonian time evolution is itself canonical}$$

$$\Downarrow$$

$$\boldsymbol{p}^T d\boldsymbol{q} - \boldsymbol{P}^T d\boldsymbol{Q} \text{ is locally exact} \rightarrow \text{generating functions in Lecture 18.}$$

Protected goals.

- Begin from the exact Lecture 16 handoff: determine which phase-space coordinate changes preserve the canonical brackets.
- Make the one-degree-of-freedom structure concrete before introducing the general symplectic form: $\{Q,P\}=1$ is exactly preservation of the oriented area form $dq \wedge dp$.
- Generalize to N degrees of freedom with $\omega = \sum_i dq_i \wedge dp_i$, emphasizing that a canonical transformation may mix conjugate pairs while preserving the full pairing structure.
- Derive the local Jacobian criterion $\mathsf{C}\mathsf{J}\mathsf{C}^T = \mathsf{J}$ from the canonical brackets, then use it to prove invariance of the Poisson bracket and Hamiltonian form.
- Distinguish symplectic preservation from Liouville volume preservation: for one degree of freedom the symplectic form is the phase-space volume form, while for $N>1$ preservation of the top-dimensional volume form is strictly weaker.
- Give explicit canonical and noncanonical transformations so the criterion is operational rather than purely formal.
- Show that Hamiltonian evolution preserves canonical brackets computed in the initial labels. For an autonomous Hamiltonian the flow is a one-parameter family; a driven flow depends on both endpoint times.
- End at the exact-differential statement that motivates generating functions; do not develop the generating-function types or time-dependent transformation machinery yet.

Notation and scope. Vectors are columns. Let

$$\boldsymbol{z} = \begin{pmatrix} \boldsymbol{q} \\ \boldsymbol{p} \end{pmatrix}, \qquad \boldsymbol{Z} = \begin{pmatrix} \boldsymbol{Q} \\ \boldsymbol{P} \end{pmatrix}, \qquad \mathsf{C} = \frac{\partial \boldsymbol{Z}}{\partial \boldsymbol{z}^T}, \qquad \mathsf{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

The Jacobian convention follows the course rule for vector-by-vector derivatives. Use ω for the symplectic two-form and reserve "volume" for the top-dimensional phase-space form. For one degree of freedom these are the same two-form; for higher dimensions they are not the same object.

Pacing. The first 42 minutes build the meaning of canonical/symplectic preservation from one conjugate pair to N degrees of freedom. The next 44 minutes make the criterion operational and separate symplectic preservation from mere volume preservation. The final 19 minutes show that Hamiltonian flow is canonical and expose the exact-differential doorway to Lecture 18.

0–10 min: Resume the Lecture 16 question: what must survive?

BOARD Start from the final question of Lecture 16:

$$\boxed{(\boldsymbol{q}, \boldsymbol{p}) \longrightarrow (\boldsymbol{Q}, \boldsymbol{P}).}$$

If both coordinate systems are to be canonical, require

$$\boxed{\{Q_i, Q_j\} = 0, \qquad \{P_i, P_j\} = 0, \qquad \{Q_i, P_j\} = \delta_{ij}.}$$

SAY This is not just a naming convention. The new variables must preserve the same conjugate pairing that made Hamilton's equations canonical in the old variables.

ASK Does a canonical transformation need to preserve ordinary Euclidean lengths or angles in phase space? No. It preserves a different geometric structure.

BOARD Recall from Lecture 16

$$\{F, G\} = \left(\frac{\partial F}{\partial \boldsymbol{z}}\right)^T \mathbb{J} \frac{\partial G}{\partial \boldsymbol{z}}.$$

The task today is to determine which coordinate changes preserve this relation.

10–24 min: One conjugate pair: canonical means oriented area preservation

BOARD Begin with one degree of freedom:

$$Q = Q(q, p), \qquad P = P(q, p).$$

Then

$$dQ = \frac{\partial Q}{\partial q}\, dq + \frac{\partial Q}{\partial p}\, dp,$$

$$dP = \frac{\partial P}{\partial q}\, dq + \frac{\partial P}{\partial p}\, dp.$$

Wedge the two differentials:

$$\begin{aligned} dQ \wedge dP &= \left(\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q}\right) dq \wedge dp \\ &= \{Q, P\}\, dq \wedge dp. \end{aligned}$$

Therefore

$$\boxed{\{Q, P\} = 1 \quad \Longleftrightarrow \quad dQ \wedge dP = dq \wedge dp.}$$

SAY In one conjugate pair, the symplectic form $dq \wedge dp$ is already the phase-space volume form. Thus preservation of the symplectic structure and preservation of the oriented Liouville area coincide in one degree of freedom.

CHECK This statement is about the volume *form*, not merely unsigned area. Hamiltonian flows are continuously connected to the identity and preserve its orientation as well as its magnitude.

BOARD Callback to Lecture 15:

$$dq_{n+1} \wedge dp_{n+1} = dq_n \wedge dp_n.$$

The discrete variational update was already showing us a canonical/symplectic map in one degree of freedom.

24–42 min: Many degrees of freedom: preserve the full conjugate pairing

BOARD For N degrees of freedom define

$$\boxed{\omega = \sum_{i=1}^{N} dq_i \wedge dp_i.}$$

In new canonical coordinates,

$$\boxed{\omega = \sum_{i=1}^{N} dQ_i \wedge dP_i.}$$

SAY Be precise about "every conjugate pair." A canonical transformation may mix the old pairs with one another. What is invariant is the *full canonical pairing structure* represented by ω, equivalently the complete set of canonical brackets.

BOARD Define

$$\boldsymbol{z} = \begin{pmatrix} \boldsymbol{q} \\ \boldsymbol{p} \end{pmatrix}, \qquad \boldsymbol{Z} = \begin{pmatrix} \boldsymbol{Q} \\ \boldsymbol{P} \end{pmatrix}, \qquad \mathsf{C} = \frac{\partial \boldsymbol{Z}}{\partial \boldsymbol{z}^T}.$$

The brackets among the new coordinate functions are encoded by

$$\frac{\partial \boldsymbol{Z}}{\partial \boldsymbol{z}^T} \mathsf{J} \left(\frac{\partial \boldsymbol{Z}}{\partial \boldsymbol{z}^T} \right)^T = \mathsf{C}\mathsf{J}\mathsf{C}^T.$$

Therefore the canonical-bracket conditions are equivalent to

$$\boxed{\mathsf{C}\mathsf{J}\mathsf{C}^T = \mathsf{J}.}$$

SAY This is the compact matrix criterion. Introduce the name only now: a smooth coordinate transformation satisfying this condition is symplectic; in the present mechanics setting it is a canonical transformation.

CHECK Do not confuse this with an orthogonal transformation. The preserved bilinear structure is antisymmetric, not Euclidean.

42–58 min: Why the symplectic condition is exactly the right one

BOARD First show that the Poisson bracket keeps its form. If $F = F(\boldsymbol{Z}(\boldsymbol{z}))$, then

$$d\boldsymbol{Z} = \frac{\partial \boldsymbol{Z}}{\partial \boldsymbol{z}^T} \, d\boldsymbol{z} = \mathsf{C} \, d\boldsymbol{z}.$$

Apply the scalar chain rule through the differential:

$$\begin{aligned} dF &= \left(\frac{\partial F}{\partial \boldsymbol{Z}} \right)^T d\boldsymbol{Z} \\ &= \left(\frac{\partial F}{\partial \boldsymbol{Z}} \right)^T \mathsf{C} \, d\boldsymbol{z} \\ &= \left(\mathsf{C}^T \frac{\partial F}{\partial \boldsymbol{Z}} \right)^T d\boldsymbol{z}. \end{aligned}$$

Comparing with

$$dF = \left(\frac{\partial F}{\partial \boldsymbol{z}} \right)^T d\boldsymbol{z}$$

gives

$$\boxed{\frac{\partial F}{\partial \boldsymbol{z}} = \mathsf{C}^T \frac{\partial F}{\partial \boldsymbol{Z}}.}$$

Likewise,

$$\frac{\partial G}{\partial \boldsymbol{z}} = \mathsf{C}^T \frac{\partial G}{\partial \boldsymbol{Z}}.$$

Hence

$$\{F,G\}_{\boldsymbol{z}} = \left(\frac{\partial F}{\partial \boldsymbol{Z}}\right)^T \mathsf{C}\mathsf{J}\mathsf{C}^T \frac{\partial G}{\partial \boldsymbol{Z}}$$
$$= \left(\frac{\partial F}{\partial \boldsymbol{Z}}\right)^T \mathsf{J}\frac{\partial G}{\partial \boldsymbol{Z}}$$
$$= \boxed{\{F,G\}_{\boldsymbol{Z}}}.$$

SAY A canonical transformation preserves the bracket itself, not merely a particular Hamiltonian or a particular trajectory.

BOARD For a time-independent transformation define

$$K(\boldsymbol{Q},\boldsymbol{P}) = H(\boldsymbol{q}(\boldsymbol{Q},\boldsymbol{P}),\boldsymbol{p}(\boldsymbol{Q},\boldsymbol{P})).$$

Then

$$\dot{\boldsymbol{Z}} = \mathsf{C}\dot{\boldsymbol{z}} = \mathsf{C}\mathsf{J}\frac{\partial H}{\partial \boldsymbol{z}}.$$

Since

$$\frac{\partial H}{\partial \boldsymbol{z}} = \mathsf{C}^T \frac{\partial K}{\partial \boldsymbol{Z}},$$

we get

$$\dot{\boldsymbol{Z}} = \mathsf{C}\mathsf{J}\mathsf{C}^T \frac{\partial K}{\partial \boldsymbol{Z}} = \boxed{\mathsf{J}\frac{\partial K}{\partial \boldsymbol{Z}}}.$$

SAY This is why the word *canonical* is deserved: Hamilton's equations retain their canonical form.

58–72 min: Examples: distort phase space without destroying canonical structure

BOARD Example 1: reciprocal scaling in one pair,

$$\boxed{Q = aq, \qquad P = \frac{p}{a}}, \qquad a \neq 0.$$

Then

$$\{Q,P\} = 1.$$

Lengths and ordinary Euclidean angles change, but the canonical area form does not.

BOARD Example 2: momentum shear,

$$\boxed{Q = q, \qquad P = p + cq.}$$

Then

$$\{Q,P\} = 1.$$

A square patch can shear into a parallelogram while remaining canonical.

BOARD Nonexample: common scaling,

$$Q = aq, \qquad P = ap,$$

for which

$$\boxed{\{Q,P\} = a^2.}$$

Except for $a = \pm 1$, this is not canonical.

ASK Which geometric property survived in the first two examples that ordinary lengths did not? The conjugate phase-space pairing.

72–86 min: Symplectic preservation versus Liouville volume preservation

BOARD The symplectic form produces the canonical top-dimensional volume form

$$\boxed{\frac{\omega^N}{N!} = dq_1 \wedge dp_1 \wedge \cdots \wedge dq_N \wedge dp_N.}$$

Therefore

$$\boxed{\text{symplectic preservation} \Longrightarrow \text{Liouville volume preservation.}}$$

SAY Now make the dimensional distinction explicit. For $N = 1$,

$$\frac{\omega^1}{1!} = \omega = dq \wedge dp,$$

so preserving the symplectic form and preserving the Liouville volume form are the same condition.

BOARD For $N > 1$, preserving ω^N does *not* force preservation of ω.

BOARD Two-degree-of-freedom counterexample:

$$Q_1 = 2q_1, \qquad Q_2 = \frac{1}{2}q_2, \qquad P_1 = p_1, \qquad P_2 = p_2.$$

Write the phase-space vectors in the order used by the matrix definition:

$$\boldsymbol{z} = \begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix}, \qquad \boldsymbol{Z} = \begin{pmatrix} Q_1 \\ Q_2 \\ P_1 \\ P_2 \end{pmatrix}.$$

Then

$$\mathsf{C} = \frac{\partial \boldsymbol{Z}}{\partial \boldsymbol{z}^T} = \begin{pmatrix} \frac{\partial Q_1}{\partial q_1} & \frac{\partial Q_1}{\partial q_2} & \frac{\partial Q_1}{\partial p_1} & \frac{\partial Q_1}{\partial p_2} \\ \frac{\partial Q_2}{\partial q_1} & \frac{\partial Q_2}{\partial q_2} & \frac{\partial Q_2}{\partial p_1} & \frac{\partial Q_2}{\partial p_2} \\ \frac{\partial P_1}{\partial q_1} & \frac{\partial P_1}{\partial q_2} & \frac{\partial P_1}{\partial p_1} & \frac{\partial P_1}{\partial p_2} \\ \frac{\partial P_2}{\partial q_1} & \frac{\partial P_2}{\partial q_2} & \frac{\partial P_2}{\partial p_1} & \frac{\partial P_2}{\partial p_2} \end{pmatrix} = \boxed{\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.}$$

Therefore

$$\boxed{\det \mathsf{C} = 2 \cdot \frac{1}{2} \cdot 1 \cdot 1 = 1,}$$

so the full phase-space volume is preserved. However,

$$\mathsf{J} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad \mathsf{C}\mathsf{J}\mathsf{C}^T = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ -2 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \end{pmatrix} \neq \mathsf{J}.$$

Equivalently,

$$\boxed{\{Q_1, P_1\} = 2, \qquad \{Q_2, P_2\} = \frac{1}{2},}$$

so the transformation is not canonical.

CHECK This is the key hierarchy:

$$\boxed{N = 1: \ \text{symplectic} = \text{Liouville volume-form preserving},}$$

while

$$\boxed{N > 1: \ \text{symplectic} \Rightarrow \text{volume preserving}, \qquad \text{volume preserving} \not\Rightarrow \text{symplectic}.}$$

86–99 min: Hamiltonian time evolution is itself canonical

BOARD Assume a smooth *autonomous* Hamiltonian and a locally unique flow. Write the evolved components as $Q_{i,\tau}(\boldsymbol{z}_0), P_{j,\tau}(\boldsymbol{z}_0)$ with $\tau = t - t_0$. All brackets in this block use the *initial canonical labels* $\boldsymbol{z}_0$; set $H_0 = H(\boldsymbol{z}_0)$.

SAY Flow composition $\Phi_\tau \circ \Phi_s = \Phi_{\tau+s}$, differentiated in s at zero, gives

$$\frac{\partial Q_{i,\tau}}{\partial \tau} = \{Q_{i,\tau}, H_0\}_{\boldsymbol{z}_0}, \qquad \frac{\partial P_{j,\tau}}{\partial \tau} = \{P_{j,\tau}, H_0\}_{\boldsymbol{z}_0}.$$

This uses autonomous flow composition, not an assumption that the map is already canonical.

BOARD Define $B_{ij}(\boldsymbol{z}_0, \tau) = \{Q_{i,\tau}, P_{j,\tau}\}_{\boldsymbol{z}_0}$. Jacobi gives

$$\begin{aligned}\frac{\partial B_{ij}}{\partial \tau} &= \{\{Q_{i,\tau}, H_0\}_{\boldsymbol{z}_0}, P_{j,\tau}\}_{\boldsymbol{z}_0} + \{Q_{i,\tau}, \{P_{j,\tau}, H_0\}_{\boldsymbol{z}_0}\}_{\boldsymbol{z}_0} \\ &= \boxed{\{B_{ij}, H_0\}_{\boldsymbol{z}_0}}.\end{aligned}$$

CHECK Put $E_{ij} = B_{ij} - \delta_{ij}$. Initially $E_{ij} = 0$ everywhere. The transport equation $\partial_\tau E_{ij} = \{E_{ij}, H_0\}_{\boldsymbol{z}_0}$ makes E_{ij} constant along backward Hamiltonian characteristics; hence it remains zero. A vanishing derivative *only at the initial time* would not suffice.

BOARD The same argument applies to the initially zero QQ and PP brackets:

$$\boxed{\{Q_{i,\tau}, P_{j,\tau}\}_{\boldsymbol{z}_0} = \delta_{ij}, \qquad \{Q_{i,\tau}, Q_{j,\tau}\}_{\boldsymbol{z}_0} = \{P_{i,\tau}, P_{j,\tau}\}_{\boldsymbol{z}_0} = 0.}$$

SAY Autonomous evolution is a one-parameter family of canonical maps. For $H(\boldsymbol{z}, t)$, each fixed-time map Φ_{t,t_0} is still symplectic, but generally depends on both times; the student notes give the Jacobian proof for that case. Liouville incompressibility follows from this stronger preserved geometry.

99–105 min: Exact one-form difference: the doorway to generating functions

BOARD Introduce the canonical one-form

$$\boxed{\vartheta = \boldsymbol{p}^T d\boldsymbol{q} = \sum_i p_i \, dq_i.}$$

BOARD Compare the ordinary and exterior product rules:

$$\frac{d}{d\lambda}\left(p_i \frac{dq_i}{d\lambda}\right) = \frac{dp_i}{d\lambda}\frac{dq_i}{d\lambda} + p_i \frac{d^2 q_i}{d\lambda^2},$$
$$d(p_i \, dq_i) = dp_i \wedge dq_i + p_i \, d^2 q_i.$$

SAY The product-rule pattern is the same. The wedge appears because dp_i and dq_i are both one-forms, whose exterior product is antisymmetric.

BOARD Therefore

$$\begin{aligned} -d\vartheta &= -\sum_i d(p_i \, dq_i) \\ &= -\sum_i \left(dp_i \wedge dq_i + p_i \, d^2 q_i\right) \\ &= \sum_i dq_i \wedge dp_i \equiv \omega, \end{aligned}$$

because $d^2 q_i = 0$ and $dp_i \wedge dq_i = -dq_i \wedge dp_i$. For a canonical transformation, preservation of ω gives

$$\boxed{d\left(\boldsymbol{p}^T d\boldsymbol{q} - \boldsymbol{P}^T d\boldsymbol{Q}\right) = 0 \quad \Longrightarrow \quad \boldsymbol{p}^T d\boldsymbol{q} - \boldsymbol{P}^T d\boldsymbol{Q} = dF \;\; \text{locally.}}$$

SAY The scalar F appears because the symplectic two-form is preserved while the corresponding canonical one-forms may differ by an exact differential.

NEXT Lecture 18 asks how choices of independent variables for F turn this relation into a practical construction method for canonical transformations.

STOP Do not introduce F_1, F_2, F_3, F_4, time-dependent canonical transformations, or the transformed-Hamiltonian correction yet. Those belong to Lecture 18.