Lecture 16 Instructor Outline — Draft

Hamiltonian Mechanics II: Poisson Brackets and Phase-Space Flow

PHYS V1100: Analytical Dynamics — 105 minutes

Evolution, conserved quantities, generators, and Liouville's theorem

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Core arc:

$$\dot{F} = \left(\frac{\partial F}{\partial \boldsymbol{q}}\right)^T \frac{\partial H}{\partial \boldsymbol{p}} - \left(\frac{\partial F}{\partial \boldsymbol{p}}\right)^T \frac{\partial H}{\partial \boldsymbol{q}} + \frac{\partial F}{\partial t}$$

$$\Downarrow \text{ name the joint operation between two functions}$$

$$\{F, G\} = \left(\frac{\partial F}{\partial \boldsymbol{q}}\right)^T \frac{\partial G}{\partial \boldsymbol{p}} - \left(\frac{\partial F}{\partial \boldsymbol{p}}\right)^T \frac{\partial G}{\partial \boldsymbol{q}}$$

$$\Downarrow \; G = H$$

$$\dot{F} = \{F, H\} + \frac{\partial F}{\partial t} \rightarrow \text{conservation under } H \rightarrow \text{generators and symmetries}$$

$$\Downarrow$$

Hamiltonian phase-space flow → Liouville incompressibility → canonical-structure handoff

Protected goals.

- Begin exactly from the unnamed antisymmetric evolution law left at the end of Lecture 15.
- Stress that the Poisson bracket is a *joint relation between two phase-space functions*; it is not a property of either function alone.
- Interpret $\{F, H\}$ as the change of F generated by the Hamiltonian flow, and state conservation as $\dot{F} = \{F, H\} + \partial F/\partial t = 0$.
- Recover Hamilton's equations from the canonical brackets $\{q_i, p_j\} = \delta_{ij}$.
- Develop antisymmetry, bilinearity, product rule, and the Jacobi identity only after the dynamical meaning of the bracket is visible.
- Connect cyclic variables and conserved quantities to the bracket language, then show that angular momentum generates rotations.
- Derive phase-space incompressibility directly from Hamilton's equations and identify it as Liouville's theorem.
- Keep energy conservation distinct from Liouville volume preservation.
- Use the compact constant matrix J only after the component structure is understood; use it to prepare canonical transformations in Lecture 17.

Notation and scope. Vectors are columns. For scalar phase-space functions, $\partial F/\partial \boldsymbol{q}$ and $\partial F/\partial \boldsymbol{p}$ are column gradients. No gradient-symbol shorthand is needed. The notation $\partial F/\partial t$ always means explicit time dependence at fixed $(\boldsymbol{q}, \boldsymbol{p})$, while $\dot{F}$ is the total derivative along the Hamiltonian trajectory.

Pacing. The first 48 minutes establish the bracket as the common language of evolution and conservation. The next 20 minutes use angular momentum to connect brackets with generators and symmetry. The final 37 minutes develop phase-space flow, Liouville's theorem, conserved-quantity algebra, compact symplectic notation, and the Lecture 17 handoff.

0–10 min: Resume the unfinished equation: one Hamiltonian, any phase-space function

BOARD Start with the exact last equation from Lecture 15. For

$$F = F(\boldsymbol{q}, \boldsymbol{p}, t),$$

Hamilton's equations give

$$\boxed{\dot{F} = \left(\frac{\partial F}{\partial \boldsymbol{q}}\right)^T \frac{\partial H}{\partial \boldsymbol{p}} - \left(\frac{\partial F}{\partial \boldsymbol{p}}\right)^T \frac{\partial H}{\partial \boldsymbol{q}} + \frac{\partial F}{\partial t}.}$$

SAY The first two terms are not special to F and H separately. They are one antisymmetric relation between *two* functions on phase space.

BOARD Define, for any two scalar phase-space functions F and G,

$$\boxed{\{F,G\} \equiv \left(\frac{\partial F}{\partial \boldsymbol{q}}\right)^T \frac{\partial G}{\partial \boldsymbol{p}} - \left(\frac{\partial F}{\partial \boldsymbol{p}}\right)^T \frac{\partial G}{\partial \boldsymbol{q}}.}$$

Therefore

$$\boxed{\dot{F} = \{F,H\} + \frac{\partial F}{\partial t}.}$$

SAY Stress the relational meaning: $\{F,G\}$ is a *joint quantity*. Once G is chosen to be the Hamiltonian, H generates the actual time flow and $\{F,H\}$ measures how F changes under that flow.

ASK Can the same function F be conserved for one Hamiltonian and not for another? Yes; conservation is a relation between F and the chosen dynamics.

10–22 min: Canonical brackets and Hamilton's equations inside one formula

BOARD For canonical phase-space coordinates,

$$\boxed{\{q_i,q_j\} = 0, \qquad \{p_i,p_j\} = 0, \qquad \{q_i,p_j\} = \delta_{ij}.}$$

BOARD Now choose $F = q_i$:

$$\dot{q}_i = \{q_i,H\} = \frac{\partial H}{\partial p_i}.$$

Choose $F = p_i$:

$$\dot{p}_i = \{p_i,H\} = -\frac{\partial H}{\partial q_i}.$$

Thus

$$\boxed{\text{Hamilton's equations are the coordinate cases of } \dot{F} = \{F,H\}.}$$

SAY The Hamiltonian is not merely one conserved scalar among others. It is the function whose bracket gives the time evolution of every phase-space observable.

BOARD Quick relational example. Let

$$F = p_x.$$

If

$$H_1 = \frac{p_x^2 + p_y^2}{2m} + V(y),$$

then

$$\boxed{\{p_x,H_1\} = 0.}$$

But for

$$H_2 = \frac{p_x^2 + p_y^2}{2m} + V(x,y),$$

generally

$$\boxed{\{p_x,H_2\} = -\frac{\partial H_2}{\partial x} \neq 0.}$$

CHECK Same F; different H; different conservation statement.

22–35 min: Algebraic structure of the Poisson bracket

BOARD Antisymmetry follows directly:

$$\boxed{\{F, G\} = -\{G, F\}.}$$

Hence

$$\boxed{\{F, F\} = 0.}$$

BOARD Bilinearity:

$$\{aF + bG, K\} = a\{F, K\} + b\{G, K\}.$$

BOARD Product rule:

$$\boxed{\{FG, K\} = F\{G, K\} + G\{F, K\}.}$$

BOARD State the Jacobi identity:

$$\boxed{\{F, \{G, K\}\} + \{G, \{K, F\}\} + \{K, \{F, G\}\} = 0.}$$

SAY Do not grind through the component proof. The structural message is that the bracket-generated transformations close consistently; this identity will matter when we compare and compose generators.

CHECK Distinguish multiplication from the bracket. $FG = GF$ for ordinary scalar functions, but the Poisson bracket is antisymmetric and need not vanish.

35–48 min: Conserved quantities are conserved relative to a Hamiltonian

BOARD The full conservation condition is

$$\boxed{\dot{F} = \{F, H\} + \frac{\partial F}{\partial t} = 0.}$$

SAY This is the correct general statement. Do not replace it too quickly by $\{F, H\} = 0$.

BOARD If F has no explicit time dependence,

$$\boxed{F \text{ conserved under } H \quad \Longleftrightarrow \quad \{F, H\} = 0.}$$

BOARD For the Hamiltonian itself,

$$\dot{H} = \{H, H\} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t},$$

so an autonomous Hamiltonian satisfies

$$\boxed{\dot{H} = 0.}$$

BOARD Cyclic coordinate:

$$\frac{\partial H}{\partial q_j} = 0 \quad \Longrightarrow \quad \dot{p}_j = \{p_j, H\} = 0.$$

Thus

$$\boxed{q_j \text{ cyclic} \Longrightarrow p_j \text{ conserved.}}$$

SAY This is the same symmetry/conservation statement seen earlier in Lagrangian mechanics, now expressed as a relation between the conjugate momentum and the Hamiltonian.

48–68 min: Generators: angular momentum produces rotations

BOARD Generalize the role played by H. Let $G(\boldsymbol{q}, \boldsymbol{p})$ generate an infinitesimal transformation with small parameter ϵ:

$$\boxed{\delta F = \epsilon\{F, G\}.}$$

Time evolution is the special case $G = H$ with $\epsilon = dt$.

SAY The bracket therefore answers a broader question: how does one phase-space function F respond to the transformation generated by another function G?

BOARD For one particle in Cartesian canonical variables,

$$\boxed{\boldsymbol{L} = \boldsymbol{x} \times \boldsymbol{p}.}$$

For rotation about z,

$$L_z = xp_y - yp_x.$$

Then

$$\boxed{\{x, L_z\} = -y, \qquad \{y, L_z\} = x,}$$

so

$$\delta x = -\epsilon y, \qquad \delta y = \epsilon x,$$

which is precisely an infinitesimal rotation.

BOARD State the general relations:

$$\boxed{\{x_i, L_j\} = \epsilon_{ijk} x_k, \qquad \{p_i, L_j\} = \epsilon_{ijk} p_k,}$$

$$\boxed{\{L_i, L_j\} = \epsilon_{ijk} L_k.}$$

SAY Angular momentum is therefore not merely associated with rotations by an external mnemonic. It is the phase-space generator of rotations.

BOARD Symmetry/conservation chain. If G generates a transformation,

$$\delta H = \epsilon\{H, G\}.$$

If the transformation is a symmetry,

$$\{H, G\} = 0.$$

By antisymmetry,

$$\{G, H\} = 0,$$

and if G has no explicit time dependence,

$$\boxed{\dot{G} = 0.}$$

SAY This is the Hamiltonian form of the Noether idea already developed earlier in the course: the generator of a continuous symmetry is conserved.

68–82 min: Hamiltonian flow and the continuum Liouville theorem

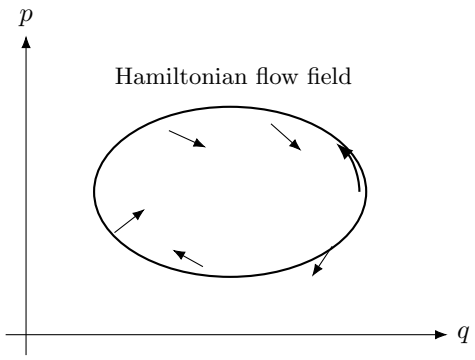
BOARD Collect the canonical coordinates into

$$\boxed{\boldsymbol{z} = \begin{pmatrix} \boldsymbol{q} \\ \boldsymbol{p} \end{pmatrix}.}$$

Hamilton's equations assign a phase-space velocity to each point:

$$\boxed{\dot{\boldsymbol{z}} = \begin{pmatrix} \partial H/\partial \boldsymbol{p} \\ -\partial H/\partial \boldsymbol{q} \end{pmatrix}.}$$

SAY This returns to the course-opening picture: a state plus an evolution law defines a flow through state space. Here the state space is canonical phase space.



BOARD Compute the phase-space divergence directly:

$$\sum_{i=1}^{N}\left(\frac{\partial \dot{q}_i}{\partial q_i}+\frac{\partial \dot{p}_i}{\partial p_i}\right).$$

Using Hamilton's equations,

$$\begin{aligned}\sum_i\left(\frac{\partial \dot{q}_i}{\partial q_i}+\frac{\partial \dot{p}_i}{\partial p_i}\right) &= \sum_i\left(\frac{\partial^2 H}{\partial q_i \partial p_i}-\frac{\partial^2 H}{\partial p_i \partial q_i}\right)\\ &= 0.\end{aligned}$$

Therefore

$$\boxed{\text{Hamiltonian flow is incompressible in canonical phase space.}}$$

BOARD A phase-space volume element

$$\boxed{d\Gamma = dq_1\, dp_1 \cdots dq_N\, dp_N}$$

is carried by the flow without changing volume:

$$\boxed{\frac{d}{dt}d\Gamma = 0.}$$

Call this Liouville's theorem.

CHECK Connect explicitly to Lecture 15: the discrete variational update preserved $\sum_i dq_i \wedge dp_i$; here the continuous Hamiltonian flow has zero phase-space divergence.

82–90 min: Liouville is not energy conservation

BOARD Keep the two conservation statements separate:

$\partial H/\partial t = 0$	$\Longrightarrow$ H conserved along each trajectory,
Hamiltonian canonical structure	$\Longrightarrow$ phase-space volume preserved by the flow.

SAY A time-dependent Hamiltonian can fail to conserve H and still generate an incompressible canonical phase-space flow.

BOARD Contrast with dissipation. A damped oscillator can spiral toward an attracting fixed point in a reduced (x, p)-like portrait. A genuine Hamiltonian flow in the full canonical phase space cannot contract an open phase-space volume into an attractor.

SAY This is another instance of an earlier course theme: reduced descriptions may look dissipative even when the larger closed description preserves more structure.

90–97 min: Conserved quantities form a Poisson algebra; involution

BOARD Suppose F and G have no explicit time dependence and

$$\{F, H\} = 0, \qquad \{G, H\} = 0.$$

Use Jacobi:

$$\{\{F, G\}, H\} + \{\{G, H\}, F\} + \{\{H, F\}, G\} = 0.$$

The last two terms vanish, so

$$\boxed{\{\{F, G\}, H\} = 0.}$$

Therefore

$$\boxed{F, G \text{ conserved} \Longrightarrow \{F, G\} \text{ conserved.}}$$

SAY Conserved quantities are not merely a list; they can have internal algebraic structure. Angular momentum provides the concrete example $\{L_i, L_j\} = \epsilon_{ijk} L_k$.

BOARD Introduce one future-use term:

$$\boxed{\{F, G\} = 0 \quad \Longrightarrow \quad F \text{ and } G \text{ are in involution.}}$$

SAY Do not develop integrability here. Flag that mutual involution will become important when we reach action-angle variables.

97–102 min: Compress the structure into the constant antisymmetric matrix

BOARD Only now introduce the phase-space matrix

$$\boxed{\mathsf{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.}$$

With

$$\boldsymbol{z} = \begin{pmatrix} \boldsymbol{q} \\ \boldsymbol{p} \end{pmatrix},$$

Hamilton's equations become

$$\boxed{\dot{\boldsymbol{z}} = \mathsf{J} \frac{\partial H}{\partial \boldsymbol{z}}.}$$

The Poisson bracket becomes

$$\boxed{\{F, G\} = \left(\frac{\partial F}{\partial \boldsymbol{z}}\right)^T \mathsf{J} \frac{\partial G}{\partial \boldsymbol{z}}.}$$

SAY This matrix is not where the structure came from; it is a compact representation of the conjugate pairing and antisymmetric phase-space geometry already made visible.

CHECK $\mathsf{J}^T = -\mathsf{J}$ immediately explains $\{F, G\} = -\{G, F\}$.

NEXT **102–105 min: canonical-transformation handoff.** Ask which changes

$$(\boldsymbol{q}, \boldsymbol{p}) \longrightarrow (\boldsymbol{Q}, \boldsymbol{P})$$

preserve the canonical relations

$$\boxed{\{Q_i, Q_j\} = 0, \qquad \{P_i, P_j\} = 0, \qquad \{Q_i, P_j\} = \delta_{ij}.}$$

A transformation that preserves this structure should preserve the form of Hamilton's equations. That is the organizing question for Lecture 17.

STOP Do not develop generating functions yet. End with: *which coordinate changes preserve the Poisson-bracket/symplectic structure?*