Lecture 15 Instructor Outline — Draft

Hamiltonian Mechanics I: Conjugate Variables and Phase Space

PHYS V1100: Analytical Dynamics — 105 minutes

Discrete action, conjugate momentum, Legendre transform, and Hamilton's equations

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Core arc:

$$\boxed{\begin{array}{c}\text{discrete action edge } \mathcal{A}_n(q_n, q_{n+1}) \to \delta\mathcal{A}_n = -p_n^- \delta q_n + p_{n+1}^+ \delta q_{n+1} \\ \Downarrow \text{ stationarity at an interior node} \\ p_n^+ = p_n^- \equiv p_n \to (q_n, p_n) \text{ as a phase-space state} \\ \Downarrow \text{ continuum / regular Legendre map} \\ \boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{q}}} \leftrightarrow \dot{\boldsymbol{q}} \to H(\boldsymbol{q}, \boldsymbol{p}, t) = \boldsymbol{p}^T \dot{\boldsymbol{q}} - L \\ \dot{\boldsymbol{q}} = \frac{\partial H}{\partial \boldsymbol{p}}, \quad \Downarrow \quad \dot{\boldsymbol{p}} = -\frac{\partial H}{\partial \boldsymbol{q}}\end{array}}$$

Protected goals.

- Return to the discrete action from Lecture 4 and use its endpoint variation to explain why momentum is *conjugate* to configuration.
- Make clear why $\dot{\boldsymbol{q}}$ and $\boldsymbol{p}$ play different geometric roles even when $\boldsymbol{p} = M\dot{\boldsymbol{q}}$ in simple coordinates.
- Show how stationarity matches the incoming and outgoing endpoint momenta and thereby lifts a configuration sequence to a phase-space sequence.
- Use the endpoint variation of the on-shell action to expose both conjugate pairs $(\boldsymbol{q}, \boldsymbol{p})$ and $(t, -H)$.
- Give a short discrete symplectic/Liouville preview: a regular variational update preserves $\sum_j dq_j \wedge dp_j$; distinguish this from energy conservation.
- Motivate the Legendre transform as the change from velocity variables to conjugate-momentum variables, not as an algebraic trick.
- Derive Hamilton's equations from the differential of H using the course vector-derivative conventions.
- Use the harmonic oscillator as the first phase-space example and central-force motion as the familiar curvilinear example.
- Stop before naming the Poisson bracket; leave its antisymmetric structure visible for Lecture 16.

Notation and scope. Vectors are columns. For a scalar function of a vector, $\partial f/\partial \boldsymbol{q}$ is a column gradient, so

$$df = \left(\frac{\partial f}{\partial \boldsymbol{q}}\right)^T d\boldsymbol{q} + \cdots .$$

For the discrete-action discussion, start with one degree of freedom to keep endpoint signs visible, then state the vector generalization. The discrete symplectic statement is a structural preview, not yet the full symplectic-geometry development planned for Lectures 17–18.

Pacing. The first 55 minutes build phase space directly from discrete variational structure. The next 27 minutes introduce the Legendre transform and derive Hamilton's equations. The final 23 minutes use familiar examples and prepare the Poisson-bracket handoff.

0–8 min: Return to the course-opening question: what is the state?

BOARD For a regular Lagrangian system,

$$\boxed{L = L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t).}$$

The Euler–Lagrange equations are second order in $\boldsymbol{q}$, so one local state description is

$$\boxed{(\boldsymbol{q}, \dot{\boldsymbol{q}}).}$$

But earlier the variational calculation produced

$$\boxed{\boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{q}}}.}$$

Today's question is

Why is $(\boldsymbol{q}, \boldsymbol{p})$ a natural state, and what does "conjugate" mean?

SAY Do not answer "because that is the definition of canonical momentum." The word *conjugate* should come from a structural pairing in the action.

ASK What information in the action distinguished $\partial L/\partial \dot{q}$ from the velocity itself when we derived the discrete Euler–Lagrange equation?

8–23 min: Recall the discrete action: one node, two neighboring edges

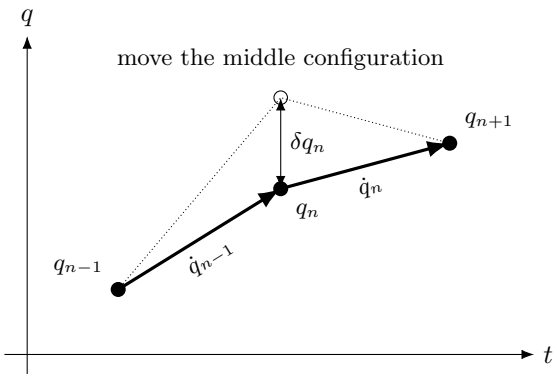
BOARD Reuse the Lecture 4 three-event geometry. With fixed spacing Δt,

$$q_{n-1}, \qquad q_n, \qquad q_{n+1},$$

with discrete slopes

$$\dot{q}_{n-1} \approx \frac{q_n - q_{n-1}}{\Delta t}, \qquad \dot{q}_n \approx \frac{q_{n+1} - q_n}{\Delta t}.$$

Varying q_n changes both adjacent slopes.



BOARD The earlier left-endpoint discrete action was

$$\mathcal{S}_d = \sum_n L\left(q_n, \frac{q_{n+1} - q_n}{\Delta t}\right) \Delta t.$$

Stationarity at q_n gave

$$\boxed{\frac{\dfrac{\partial L_n}{\partial \dot{q}_n} - \dfrac{\partial L_{n-1}}{\partial \dot{q}_{n-1}}}{\Delta t} = \frac{\partial L_n}{\partial q_n}.}$$

This was where generalized momentum first appeared:

$$\boxed{p_n = \frac{\partial L_n}{\partial \dot{q}_n}.}$$

SAY Today we recast the same local variation in a way that makes the endpoint pairing visible before taking a continuum limit.

23–35 min: One action edge generates momentum at its endpoints

BOARD Write one discrete action contribution abstractly as

$$\boxed{\mathcal{A}_n = \mathcal{A}_n(q_n, q_{n+1}).}$$

Its variation is

$$\delta\mathcal{A}_n = \frac{\partial \mathcal{A}_n}{\partial q_n}\,\delta q_n + \frac{\partial \mathcal{A}_n}{\partial q_{n+1}}\,\delta q_{n+1}.$$

Define the endpoint momenta

$$\boxed{p_n^- = -\frac{\partial \mathcal{A}_n}{\partial q_n}, \qquad p_{n+1}^+ = \frac{\partial \mathcal{A}_n}{\partial q_{n+1}}.}$$

Then

$$\boxed{\delta\mathcal{A}_n = -p_n^-\,\delta q_n + p_{n+1}^+\,\delta q_{n+1}.}$$

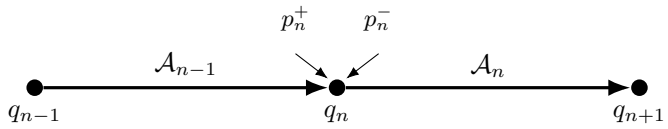
SAY This is the first place to attach the word *conjugate*: momentum is the coefficient paired with an endpoint configuration variation.

BOARD At the shared node q_n, two neighboring edges contribute

$$\delta(\mathcal{A}_{n-1} + \mathcal{A}_n) \supset \left(p_n^+ - p_n^-\right)\delta q_n.$$

Stationarity for arbitrary δq_n gives

$$\boxed{p_n^+ = p_n^- \equiv p_n.}$$



$p_n^+ = p_n^- \equiv p_n$ on a stationary path

CHECK For the particular left-endpoint discretization used in Lecture 4, this matching condition is exactly the earlier discrete Euler–Lagrange equation. Do not identify p_n^- and p_n^+ separately with the same $\partial L/\partial \dot{q}$ off shell; the matching statement is the invariant content.

35–47 min: What "conjugate" means, and why $\dot{q}$ is not the conjugate variable

BOARD For many coordinates, the endpoint term is

$$\boxed{\delta\mathcal{A} = -\boldsymbol{p}_i^T\delta\boldsymbol{q}_i + \boldsymbol{p}_f^T\delta\boldsymbol{q}_f.}$$

Thus the natural scalar pairing is

$$\boxed{\boldsymbol{p}^T\delta\boldsymbol{q}.}$$

SAY Momentum is conjugate to configuration because it is the dual quantity naturally paired with a configuration displacement in the variation of the action.

BOARD Contrast the roles:

$$\boxed{\dot{\boldsymbol{q}}:\ \text{tangent / velocity-like object}, \qquad \boldsymbol{p}:\ \text{dual / momentum-like object.}}$$

For a natural mechanical system,

$$T = \frac{1}{2}\dot{\boldsymbol{q}}^T M(\boldsymbol{q})\dot{\boldsymbol{q}}, \qquad \boxed{\boldsymbol{p} = M(\boldsymbol{q})\dot{\boldsymbol{q}}.}$$

The mass matrix supplies the map from velocity to momentum.

BOARD Polar-coordinate example:

$$T = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2),$$

so

$$\boxed{p_r = m\dot{r}, \qquad p_\theta = mr^2\dot{\theta} = J.}$$

ASK What is conjugate to the angle θ: angular velocity or angular momentum?

CHECK The answer is $p_\theta = J$. Cartesian $p = m\dot{q}$ can hide the distinction because the metric/mass map is so simple there.

47–55 min: Time as a coordinate: the conjugate pair $(t, -H)$

BOARD For an extremal continuum action segment whose endpoint configurations and endpoint times may vary,

$$\boxed{\delta S_{\text{on shell}} = \left[\boldsymbol{p}^T \delta \boldsymbol{q} - H\, \delta t\right]_i^f.}$$

Therefore the two endpoint pairings are

$$\boxed{\boldsymbol{q} \longleftrightarrow \boldsymbol{p}, \qquad t \longleftrightarrow -H.}$$

SAY This gives the Hamiltonian a structural meaning before we define it by a Legendre transform: $-H$ is the quantity conjugate to time in the endpoint variation of the action.

BOARD Symmetry/conservation parallel:

$$\boxed{q_j \text{ absent} \implies p_j \text{ conserved},}$$

$$\boxed{t \text{ absent} \implies H \text{ conserved}.}$$

SAY Phrase this carefully: the *variables* come in conjugate pairs. They are not generally both conserved. Translation symmetry in one member implies conservation of the conjugate momentum.

CHECK For a fully discrete exact time-translation statement, the time nodes must participate in the variational principle (or one must use the appropriate discrete energy). A fixed-step discretization need not conserve the continuum H exactly.

55–62 min: Discrete phase-space map and a Liouville/symplectic preview

BOARD Stationarity has lifted the configuration sequence to

$$\boxed{(q_n, p_n) \longmapsto (q_{n+1}, p_{n+1}).}$$

For one regular discrete action edge,

$$d\mathcal{A}_n = -p_n\, dq_n + p_{n+1}\, dq_{n+1}.$$

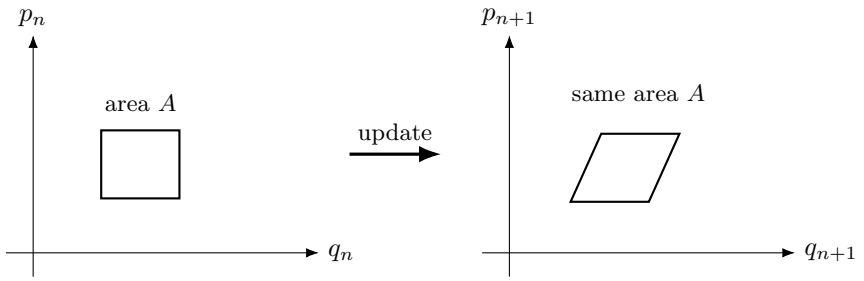
Take one more exterior differential:

$$0 = d^2\mathcal{A}_n = -dp_n \wedge dq_n + dp_{n+1} \wedge dq_{n+1}.$$

Hence

$$\boxed{dq_{n+1} \wedge dp_{n+1} = dq_n \wedge dp_n.}$$

SAY A small phase-space area element may shear or stretch, but its oriented area is preserved by the variational update. This is the one-degree-of-freedom discrete symplectic statement and the precursor of Liouville's theorem.



BOARD For N degrees of freedom, preview

$$\boxed{\omega = \sum_{j=1}^{N} dq_j \wedge dp_j \quad \text{is preserved.}}$$

CHECK Keep two statements distinct:

time-translation symmetry	$\Rightarrow H$ conservation,
Hamiltonian / symplectic structure	$\Rightarrow$ phase-space volume preservation.

The second does not require H to be time independent in the same way the first does.

62–72 min: The Legendre transform: change from velocity to conjugate momentum

BOARD Return to the continuum Lagrangian

$$L = L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t), \qquad \boxed{\boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{q}}}.}$$

If the map $\dot{\boldsymbol{q}} \mapsto \boldsymbol{p}$ is locally invertible, write

$$\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}(\boldsymbol{q}, \boldsymbol{p}, t).$$

Define

$$\boxed{H(\boldsymbol{q}, \boldsymbol{p}, t) = \boldsymbol{p}^T \dot{\boldsymbol{q}} - L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t).}$$

SAY The Legendre transform is the operation that changes which variable is independent: from the tangent/velocity description $(\boldsymbol{q}, \dot{\boldsymbol{q}})$ to the conjugate/dual description $(\boldsymbol{q}, \boldsymbol{p})$.

BOARD Regularity condition:

$$\boxed{\frac{\partial^2 L}{\partial \dot{\boldsymbol{q}}\, \partial \dot{\boldsymbol{q}}^T} \text{ invertible.}}$$

For the present course, stay with regular systems.

OPTIONAL If this Hessian is singular, the velocity-to-momentum map is not invertible without additional constraint structure. Mention constrained Hamiltonian systems only as a boundary of scope.

72–84 min: Derive Hamilton's equations from the differential of H

BOARD Differentiate before substituting:

$$\begin{aligned} dH &= d(\boldsymbol{p}^T \dot{\boldsymbol{q}}) - dL \\ &= \dot{\boldsymbol{q}}^T d\boldsymbol{p} + \boldsymbol{p}^T d\dot{\boldsymbol{q}} - \left[\left(\frac{\partial L}{\partial \boldsymbol{q}} \right)^T d\boldsymbol{q} + \left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}} \right)^T d\dot{\boldsymbol{q}} + \frac{\partial L}{\partial t}\, dt \right]. \end{aligned}$$

Use

$$\boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{q}}}$$

to cancel the $d\dot{\boldsymbol{q}}$ terms:

$$\boxed{dH = \dot{\boldsymbol{q}}^{\,T} d\boldsymbol{p} - \left(\frac{\partial L}{\partial \boldsymbol{q}}\right)^{T} d\boldsymbol{q} - \frac{\partial L}{\partial t}\, dt.}$$

BOARD But also

$$\boxed{dH = \left(\frac{\partial H}{\partial \boldsymbol{q}}\right)^{T} d\boldsymbol{q} + \left(\frac{\partial H}{\partial \boldsymbol{p}}\right)^{T} d\boldsymbol{p} + \frac{\partial H}{\partial t}\, dt.}$$

Therefore

$$\boxed{\frac{\partial H}{\partial \boldsymbol{p}} = \dot{\boldsymbol{q}}, \qquad \frac{\partial H}{\partial \boldsymbol{q}} = -\frac{\partial L}{\partial \boldsymbol{q}}, \qquad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}.}$$

BOARD Euler–Lagrange gives

$$\dot{\boldsymbol{p}} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right) = \frac{\partial L}{\partial \boldsymbol{q}}.$$

Hence

$$\boxed{\dot{\boldsymbol{q}} = \frac{\partial H}{\partial \boldsymbol{p}}, \qquad \dot{\boldsymbol{p}} = -\frac{\partial H}{\partial \boldsymbol{q}}.}$$

SAY This is the main derivation. The first equation reconstructs velocity from momentum; the second is the momentum evolution law.

84–92 min: Phase space and the harmonic oscillator

BOARD Hamiltonian phase space has local coordinates

$$\boxed{(\boldsymbol{q}, \boldsymbol{p})}$$

and dimension $2N$ for N configuration degrees of freedom. Hamilton's equations define a first-order state-space flow:

$$\boxed{\frac{d}{dt}\begin{pmatrix}\boldsymbol{q}\\ \boldsymbol{p}\end{pmatrix} = \begin{pmatrix}\partial H/\partial \boldsymbol{p}\\ -\partial H/\partial \boldsymbol{q}\end{pmatrix}.}$$

SAY This returns directly to the opening language of the course: state → evolution law → trajectory through state space.

BOARD Harmonic oscillator:

$$L = \frac{1}{2} m\dot{q}^2 - \frac{1}{2} k q^2, \qquad p = m\dot{q},$$

$$\boxed{H(q,p) = \frac{p^2}{2m} + \frac{1}{2} k q^2.}$$

Hamilton's equations are

$$\boxed{\dot{q} = \frac{p}{m}, \qquad \dot{p} = -kq.}$$

Therefore

$$\ddot{q} + \frac{k}{m} q = 0.$$

BOARD Constant-H curves are ellipses:

$$\boxed{\frac{p^2}{2m} + \frac{1}{2} k q^2 = E.}$$

SAY The physical motion is unchanged; the representation exposes a first-order flow in the canonical variables. This reconnects with the phase-plane discussion from Lecture 12, but the vertical coordinate is now canonical momentum, not generically velocity.

92–101 min: Central force: the conjugate to an angle is angular momentum

BOARD Reuse the reduced planar central-force Lagrangian:

$$L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - V(r).$$

Then

$$\boxed{p_r = \mu\dot{r}, \qquad p_\theta = \mu r^2\dot{\theta} = J.}$$

The Hamiltonian is

$$\boxed{H = \frac{p_r^2}{2\mu} + \frac{p_\theta^2}{2\mu r^2} + V(r).}$$

Hamilton's equations give

$$\boxed{\dot{r} = \frac{p_r}{\mu}, \qquad \dot{\theta} = \frac{p_\theta}{\mu r^2},}$$

$$\boxed{\dot{p}_r = \frac{p_\theta^2}{\mu r^3} - \frac{dV}{dr}, \qquad \dot{p}_\theta = 0.}$$

Thus

$$H = \frac{p_r^2}{2\mu} + V_{\text{eff}}(r), \qquad \boxed{V_{\text{eff}}(r) = V(r) + \frac{J^2}{2\mu r^2}.}$$

SAY The old central-force reduction has not disappeared. Hamiltonian mechanics reorganizes it so that the conserved angular momentum is literally the canonical momentum conjugate to the cyclic angle.

101–105 min: Hamiltonian versus energy, and the Lecture 16 handoff

BOARD Definition:

$$\boxed{H = \boldsymbol{p}^T\dot{\boldsymbol{q}} - L.}$$

For the usual natural mechanical system $L = T - V$ with quadratic kinetic energy,

$$\boxed{H = T + V = E.}$$

But $H = E$ is a consequence for this class of systems, not the definition.

BOARD Along Hamiltonian motion,

$$\frac{dH}{dt} = \left(\frac{\partial H}{\partial \boldsymbol{q}}\right)^T \dot{\boldsymbol{q}} + \left(\frac{\partial H}{\partial \boldsymbol{p}}\right)^T \dot{\boldsymbol{p}} + \frac{\partial H}{\partial t}.$$

Using Hamilton's equations, the first two terms cancel:

$$\boxed{\dot{H} = \frac{\partial H}{\partial t}.}$$

Thus an autonomous Hamiltonian satisfies $\dot{H} = 0$.

NEXT For any observable $f(\boldsymbol{q}, \boldsymbol{p}, t)$,

$$\dot{f} = \left(\frac{\partial f}{\partial \boldsymbol{q}}\right)^T \dot{\boldsymbol{q}} + \left(\frac{\partial f}{\partial \boldsymbol{p}}\right)^T \dot{\boldsymbol{p}} + \frac{\partial f}{\partial t}.$$

Substitute Hamilton's equations but do *not* name the resulting structure yet:

$$\boxed{\dot{f} = \left(\frac{\partial f}{\partial \boldsymbol{q}}\right)^T \frac{\partial H}{\partial \boldsymbol{p}} - \left(\frac{\partial f}{\partial \boldsymbol{p}}\right)^T \frac{\partial H}{\partial \boldsymbol{q}} + \frac{\partial f}{\partial t}.}$$

STOP Leave the antisymmetric structure visible. Lecture 16 will compress it into the Poisson bracket and use it to discuss phase-space flow and conserved quantities.