Lecture 14 Instructor Outline — Draft

Rigid Bodies II: Dynamics, Symmetry, and the Heavy Top

PHYS V1100: Analytical Dynamics — 105 minutes

Euler equations, stability, symmetry reduction, and orientation representations

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Core arc:

$$\dot{\boldsymbol{J}}_s = \boldsymbol{\tau}_s \to \boldsymbol{\tau}_b = \dot{\boldsymbol{J}}_b + \boldsymbol{\omega}_b \times \boldsymbol{J}_b \to \mathsf{I}_b\dot{\boldsymbol{\omega}}_b + \boldsymbol{\omega}_b \times (\mathsf{I}_b\boldsymbol{\omega}_b) = \boldsymbol{\tau}_b$$

$$\Downarrow$$

principal axes $\to$ torque-free invariants $\to$ stability $\to$ symmetric top

$\Downarrow$ beyond Exam II cutoff

$Q \to Q = Q_{\boldsymbol{n}} R_3(\chi) \to (\boldsymbol{n}, \text{spin}) \to$ heavy-top reduction and quadrature

Protected goals.

- Begin from the rotating-frame derivative established in Lecture 13; do not rederive rigid-body kinematics.
- Derive Euler's vector equation before choosing principal axes, then obtain the component Euler equations as a basis choice.
- Use torque-free conservation and linearization to explain principal-axis stability and the intermediate-axis instability.
- Solve the torque-free symmetric top in body components and interpret its space-frame precession about fixed angular momentum.
- Mark the Exam II rigid-body boundary after the torque-free symmetric top; PSet6 Q15–Q16 cover this required scope.
- After the cutoff, compare orientation representations: Q as the direct frame map, a rotor as a compact geometric representation, and Euler angles as one local decomposition/coordinate choice for Q.
- Use symmetry to motivate $Q = Q_{\boldsymbol{n}} R_3(\chi)$ for the heavy symmetric top, and derive the standard one-dimensional nutation quadrature without requiring a full three-angle Euler treatment.

Notation and scope. Subscripts s, b denote space/laboratory and body components of the same physical vector. The body inertia tensor I_b is constant. In the heavy-top section the pivot is fixed, $\boldsymbol{n} = Q\boldsymbol{e}_3$ is the space direction of the body symmetry axis, and $\boldsymbol{e}_z$ is the upward laboratory vertical. The gravitational potential is therefore $V = Mg\ell\, \boldsymbol{e}_z \cdot \boldsymbol{n}$.

0–12 min: Angular-momentum balance in a moving body frame

BOARD Start exactly where Lecture 13 stopped:

$$\boxed{\dot{\boldsymbol{J}}_s = \boldsymbol{\tau}_s,} \qquad \boxed{Q^T\dot{\boldsymbol{J}}_s = \dot{\boldsymbol{J}}_b + \boldsymbol{\omega}_b \times \boldsymbol{J}_b.}$$

Therefore

$$\boxed{\boldsymbol{\tau}_b = \dot{\boldsymbol{J}}_b + \boldsymbol{\omega}_b \times \boldsymbol{J}_b.}$$

SAY In the inertial frame the balance law is simple. In the body frame the mass distribution and inertia tensor are simple. The cross-product term is the price of using both advantages at once.

BOARD Since Lecture 13 gave

$$\boxed{\boldsymbol{J}_b = \mathsf{I}_b\boldsymbol{\omega}_b}$$

and I_b is constant in body components,

$$\boxed{\mathsf{I}_b\dot{\boldsymbol{\omega}}_b + \boldsymbol{\omega}_b \times (\mathsf{I}_b\boldsymbol{\omega}_b) = \boldsymbol{\tau}_b.}$$

Call this Euler's rigid-body equation in vector form.

CHECK If $\mathsf{I}_b = II_3$ is isotropic, then $\boldsymbol{\omega} \times (\mathsf{I}\boldsymbol{\omega}) = I\boldsymbol{\omega} \times \boldsymbol{\omega} = 0$, so the equation reduces to $I\dot{\boldsymbol{\omega}} = \boldsymbol{\tau}$.

12–25 min: Principal axes and Euler's component equations

BOARD Because I_b is real and symmetric, choose an orthonormal eigenbasis:

$$\boxed{\mathsf{I}_b = \mathrm{diag}(I_1, I_2, I_3).}$$

Then

$$\boldsymbol{J}_b = \begin{pmatrix} I_1\omega_1 \\ I_2\omega_2 \\ I_3\omega_3 \end{pmatrix}.$$

BOARD Expanding the vector equation gives

$$\boxed{I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 = \tau_1,}$$

$$\boxed{I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1 = \tau_2,}$$

$$\boxed{I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 = \tau_3.}$$

SAY This is another use of the Lecture 12 symmetric-matrix logic: choose the orthogonal eigenbasis of a quadratic form. But diagonal inertia does *not* diagonalize the nonlinear dynamics; the angular-velocity components remain coupled.

CHECK A principal-axis rotation $\boldsymbol{\omega} = \Omega\boldsymbol{e}_k$ makes the quadratic coupling vanish and is therefore an exact torque-free solution.

25–40 min: Torque-free motion: conserved quantities before solutions

BOARD Set

$$\boxed{\boldsymbol{\tau} = \mathbf{0}.}$$

Then in the laboratory frame

$$\boxed{\boldsymbol{J}_s = \text{constant}.}$$

Hence

$$\boxed{J^2 = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 = \text{constant}.}$$

BOARD Rotational kinetic energy is also conserved:

$$\boxed{E = T_{\mathrm{rot}} = \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right).}$$

CHECK Verify energy conservation directly from Euler's vector equation:

$$\frac{dT}{dt} = \boldsymbol{\omega}\cdot\mathsf{I}\dot{\boldsymbol{\omega}} = -\boldsymbol{\omega}\cdot[\boldsymbol{\omega}\times(\mathsf{I}\boldsymbol{\omega})] = 0.$$

SAY In body-$\boldsymbol{\omega}$ space, torque-free motion is confined to the intersection of the constant-energy ellipsoid and the constant-J^2 ellipsoid. We will use this geometry qualitatively; a full Poinsot construction is not required.

40–55 min: Steady principal-axis rotation and the intermediate-axis instability

BOARD Assume

$$\boxed{I_1 < I_2 < I_3.}$$

Every principal-axis rotation $\boldsymbol{\omega} = \Omega\boldsymbol{e}_k$ is an exact torque-free solution. Ask whether a small transverse perturbation remains small.

BOARD About axis 1, write

$$\omega_1 \simeq \Omega, \qquad \omega_2 = \eta_2, \qquad \omega_3 = \eta_3.$$

Linearized Euler equations give

$$\dot{\eta}_2 = \frac{I_3 - I_1}{I_2}\Omega\eta_3, \qquad \dot{\eta}_3 = -\frac{I_2 - I_1}{I_3}\Omega\eta_2,$$

so

$$\boxed{\ddot{\eta}_2 = -\Omega^2 \frac{(I_3 - I_1)(I_2 - I_1)}{I_2 I_3}\eta_2.}$$

Axis 1 is linearly stable.

BOARD About axis 2,

$$\boxed{\ddot{\eta}_1 = +\Omega^2 \frac{(I_3 - I_2)(I_2 - I_1)}{I_1 I_3}\eta_1,}$$

so perturbations grow exponentially. Axis 2 is unstable.

BOARD About axis 3 the corresponding coefficient is negative, so axis 3 is stable.

smallest and largest principal moments: stable	intermediate moment: unstable.

SAY This is the Lecture 12 stability move again: linearize about an exact solution, then let the sign of the perturbation equation decide local behavior.

CHECK Demonstrate with a book or rectangular object and identify the three principal-moment axes before tossing it.

55–68 min: Torque-free symmetric top: exact reduction by symmetry

BOARD Let

$$\boxed{I_1 = I_2 = I_\perp, \qquad I_3 = I_\parallel.}$$

Euler's equations with zero torque give

$$\boxed{\dot{\omega}_3 = 0}$$

and

$$\boxed{\dot{\omega}_1 = -\Omega_b\omega_2, \qquad \dot{\omega}_2 = \Omega_b\omega_1,}$$

where

$$\boxed{\Omega_b = \frac{I_\parallel - I_\perp}{I_\perp}\,\omega_3.}$$

Thus the transverse body components rotate uniformly:

$$\omega_1 + i\omega_2 = Ce^{i\Omega_b t}.$$

SAY In body coordinates the angular-velocity components precess around the symmetry axis. In space, the torque-free angular momentum vector $\boldsymbol{J}_s$ is fixed.

BOARD Let

$$\boxed{\boldsymbol{n} = Q\boldsymbol{e}_3}$$

be the body symmetry axis in laboratory components. Kinematics gives

$$\dot{\boldsymbol{n}} = \boldsymbol{\omega}_s \times \boldsymbol{n}.$$

For a symmetric body,

$$\boldsymbol{J}_s = I_\perp\boldsymbol{\omega}_s + (I_\parallel - I_\perp)(\boldsymbol{\omega}_s \cdot \boldsymbol{n})\boldsymbol{n}.$$

Cross with $\boldsymbol{n}$:

$$\boldsymbol{J}_s \times \boldsymbol{n} = I_\perp\boldsymbol{\omega}_s \times \boldsymbol{n} = I_\perp\dot{\boldsymbol{n}}.$$

Therefore

$$\boxed{\dot{\boldsymbol{n}} = \frac{1}{I_\perp}\boldsymbol{J}_s \times \boldsymbol{n}.}$$

The symmetry axis precesses uniformly about the fixed $\boldsymbol{J}_s$ with space-frame rate $J/I_\perp$.

STOP **Exam II rigid-body boundary.** Required rigid-body material ends here: Lecture 13 kinematics plus Euler equations, principal axes, torque-free invariants, intermediate-axis stability, and the torque-free symmetric top. PSet6 Q15–Q16 provide the corresponding study problems. The remainder of this lecture is conceptual extension and is not part of Exam II.

68–78 min: Orientation representations: choose a description for a purpose

BOARD The orientation itself is the map

$$\boxed{Q \in SO(3).}$$

Three common descriptions emphasize different structures.

Description	Advantage	Tradeoff
Q	Direct frame map; acts immediately on points and axes; globally nonsingular	Nine entries with orthogonality constraints
Euler coordinates	Three local scalar generalized coordinates; cyclic coordinates can make reduction transparent	Convention dependent; coordinate singularities; angle rates are not angular-velocity components
Rotor $\mathcal{R}$	Compact geometric composition; no coordinate singularity; natural generator/bivector relation	Unit constraint, double-cover sign, and less familiar notation

SAY Euler angles are not a different physical orientation object. They are one local coordinate choice or factorization of the same Q. Their value is greatest when scalar generalized coordinates expose symmetries or cyclic variables.

OPTIONAL Rotor reminder only:

$$\boldsymbol{a}_s = \mathcal{R}\boldsymbol{a}_b\widetilde{\mathcal{R}}, \qquad \mathcal{R}\widetilde{\mathcal{R}} = 1, \qquad \mathcal{R} \sim -\mathcal{R}.$$

Do not repeat the Lecture 3/13 GA development here.

78–88 min: Symmetry suggests the decomposition of orientation

BOARD For a symmetric top, the distinguished physical direction is

$$\boxed{\boldsymbol{n} = Q\boldsymbol{e}_3.}$$

Instead of choosing three generic angles immediately, factor the orientation as

$$\boxed{Q = Q_{\boldsymbol{n}} R_3(\chi),}$$

where

$$Q_{\boldsymbol{n}}\boldsymbol{e}_3 = \boldsymbol{n}, \qquad R_3(\chi)\boldsymbol{e}_3 = \boldsymbol{e}_3.$$

SAY The decomposition separates the direction of the symmetry axis from rotation about that axis:

$$\boxed{\text{orientation} = \text{axis direction} + \text{spin about the axis.}}$$

BOARD The split is not unique:

$$Q_{\boldsymbol{n}} \to Q_{\boldsymbol{n}} R_3(\alpha), \qquad \chi \to \chi - \alpha.$$

Only the product Q is physical. An Euler convention is one particular local choice of $Q_{\boldsymbol{n}}$ together with the residual angle χ.

CHECK For example, one may choose locally

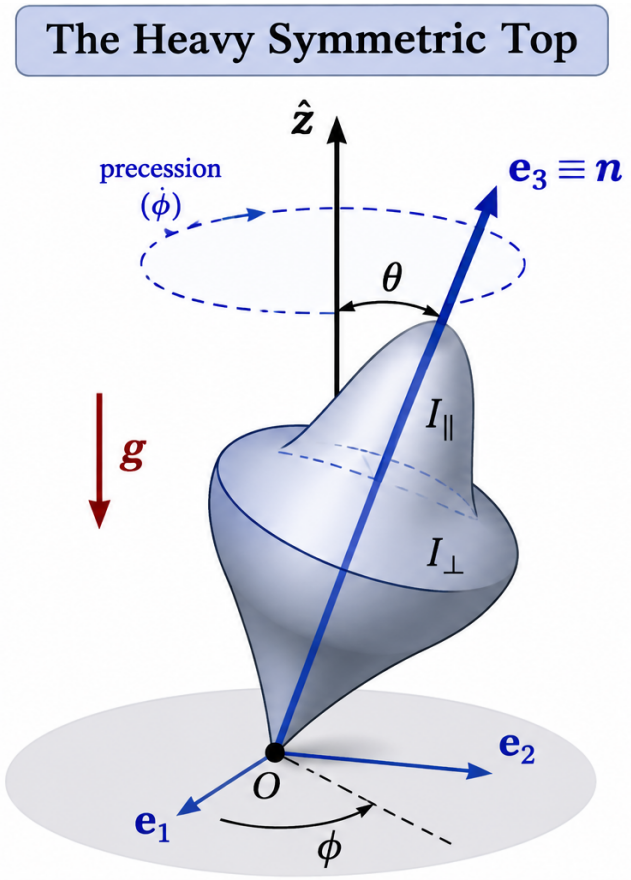
$$Q_{\boldsymbol{n}} = R_z(\phi)R_y(\theta),$$

so that

$$Q = R_z(\phi)R_y(\theta)R_3(\chi).$$

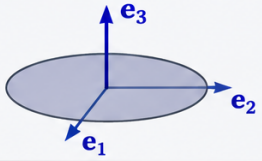
This is an Euler-type decomposition, obtained *after* identifying the symmetry-adapted variables.

88–100 min: Heavy symmetric top: reduce to the symmetry axis and two constants



Symmetric top: moments of inertia

$I_1 = I_2 \equiv I_\perp,$
$I_3 \equiv I_\parallel.$



Notation and geometry

$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$	body-fixed orthonormal axes
$\mathbf{e}_3 \equiv \boldsymbol{n}$	symmetry axis (unit vector)
$I_\perp$	moment of inertia about $\mathbf{e}_1, \mathbf{e}_2$
$I_\parallel$	moment of inertia about $\mathbf{e}_3$
θ	tilt angle: $\cos\theta = \boldsymbol{n} \cdot \hat{\mathbf{z}}$
ϕ	azimuthal (precession) angle about $\hat{\mathbf{z}}$
O	fixed point (pivot)
m	mass, with center of mass a distance ℓ
$\boldsymbol{g}$	gravitational acceleration (downward)

Key conserved quantities (heavy top)

- Symmetry about $\mathbf{e}_3$: $J_3 = \mathbf{J} \cdot \mathbf{e}_3 = I_\parallel \omega_3$ (conserved)
- Rotational symmetry about $\hat{\mathbf{z}}$: $J_z = \mathbf{J} \cdot \hat{\mathbf{z}}$ (conserved)
- Energy (no explicit t dependence): $E = T + V$ (conserved)
  $V = mg\ell\cos\theta.$

SAY Use the figure to establish the geometry before reducing the dynamics: $\boldsymbol{n} = Q\boldsymbol{e}_3$ is the body symmetry axis written in space components, $I_\perp$ is the moment about either transverse body axis, and $I_\parallel$ is the moment about the symmetry axis. The tilt θ and azimuth ϕ describe the direction of $\boldsymbol{n}$ relative to the fixed vertical.

BOARD Fix the pivot. Let the center of mass lie a distance ℓ along the body symmetry axis. Then

$$\boxed{\boldsymbol{n} = Q\boldsymbol{e}_3, \qquad \boldsymbol{r}_{\mathrm{CM}} = \ell\boldsymbol{n}.}$$

Take $\boldsymbol{e}_z$ upward, so

$$\boxed{V = Mg\ell\, \boldsymbol{e}_z \cdot \boldsymbol{n}.}$$

Use principal moments about the pivot,

$$\boxed{\mathsf{I}_O = \mathrm{diag}(I_\perp, I_\perp, I_\parallel).}$$

OPTIONAL If inertia about the center of mass is known, the pivot tensor follows from the parallel-axis theorem:

$$\mathsf{I}_O = \mathsf{I}_{\mathrm{CM}} + M\ell^2(I_3 - \boldsymbol{e}_3\boldsymbol{e}_3^T).$$

BOARD Since

$$\dot{\boldsymbol{n}} = \boldsymbol{\omega} \times \boldsymbol{n},$$

the angular velocity can be decomposed without Euler angles:

$$\boxed{\boldsymbol{\omega} = \boldsymbol{n} \times \dot{\boldsymbol{n}} + \sigma\boldsymbol{n}, \qquad \sigma = \boldsymbol{\omega} \cdot \boldsymbol{n}.}$$

Therefore

$$\boxed{T = \frac{1}{2}I_\perp|\dot{\boldsymbol{n}}|^2 + \frac{1}{2}I_\parallel\sigma^2.}$$

BOARD Rotation about the body symmetry axis gives the conserved quantity

$$\boxed{J_3 = I_\parallel \sigma = \text{constant}.}$$

Vertical rotational symmetry gives

$$\boxed{J_z = \boldsymbol{e}_z \cdot \boldsymbol{J} = \text{constant}.}$$

Moreover

$$\boxed{\boldsymbol{J} = I_\perp (\boldsymbol{n} \times \dot{\boldsymbol{n}}) + J_3 \boldsymbol{n}.}$$

BOARD Now introduce coordinates only on the sphere of possible axis directions:

$$\boldsymbol{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta).$$

Then

$$|\dot{\boldsymbol{n}}|^2 = \dot\theta^2 + \sin^2\theta\,\dot\phi^2,$$

and

$$(\boldsymbol{n} \times \dot{\boldsymbol{n}}) \cdot \boldsymbol{e}_z = \sin^2\theta\,\dot\phi.$$

Thus

$$\boxed{J_z = I_\perp \sin^2\theta\,\dot\phi + J_3\cos\theta,}$$

so

$$\boxed{\dot\phi = \frac{J_z - J_3\cos\theta}{I_\perp \sin^2\theta}.}$$

BOARD Energy is

$$E = \frac{1}{2} I_\perp \left(\dot\theta^2 + \sin^2\theta\dot\phi^2\right) + \frac{J_3^2}{2I_\parallel} + Mg\ell\cos\theta.$$

Eliminate $\dot\phi$:

$$\boxed{\frac{1}{2} I_\perp \dot\theta^2 + V_{\text{eff}}(\theta) = E - \frac{J_3^2}{2I_\parallel},}$$

with

$$\boxed{V_{\text{eff}}(\theta) = Mg\ell\cos\theta + \frac{(J_z - J_3\cos\theta)^2}{2I_\perp \sin^2\theta}.}$$

SAY This is the standard heavy-top nutation quadrature, obtained without parameterizing the full orientation by three Euler angles. Only the direction $\boldsymbol{n} \in S^2$ needed the two coordinates (θ, ϕ); the residual spin was already represented by the conserved J_3.

100–105 min: Euler angles reappear as one reconstruction choice

BOARD If we choose the local section

$$Q_{\boldsymbol{n}} = R_z(\phi) R_y(\theta)$$

and write

$$Q = Q_{\boldsymbol{n}} R_3(\chi),$$

then the physical spin rate along the symmetry axis is

$$\boxed{\sigma = \boldsymbol{\omega} \cdot \boldsymbol{n} = \dot\chi + \dot\phi\cos\theta.}$$

SAY The familiar Euler-angle combination is now geometric rather than mnemonic: the spin about $\boldsymbol{n}$ equals the explicit residual rotation rate plus the rotation about $\boldsymbol{n}$ induced by moving the chosen frame $Q_{\boldsymbol{n}}$ over the sphere.

BOARD Final representation map:

$$
\boxed{
\begin{array}{c}
Q: \text{ full orientation and direct frame map} \\
\Downarrow \text{ use symmetry} \\
Q = Q_{\boldsymbol{n}} R_3(\chi): \text{ axis direction + residual spin} \\
\Downarrow \text{ choose local coordinates only when useful} \\
\boldsymbol{n} \leftrightarrow (\theta, \phi), \qquad \chi \text{ reconstructs the full orientation}
\end{array}
}
$$

NEXT After Exam II, Hamiltonian mechanics will systematize ideas already visible here: cyclic variables, conserved momenta, reduction, and effective one-dimensional dynamics.

Pacing and stopping rule. Protect the vector Euler equation, principal-axis equations, torque-free invariants, intermediate-axis stability, and torque-free symmetric top before the Exam II boundary. After the boundary, protect the conceptual statement that Euler angles are one decomposition choice for Q, the symmetry-adapted factorization $Q = Q_{\boldsymbol{n}} R_3(\chi)$, and the heavy-top effective potential. If time is short, shorten the conserved-ellipsoid discussion, the representation table, the gauge-freedom comment, and the parallel-axis aside before cutting those structural steps.