Lecture 9 Instructor Outline — Draft

Continuum Systems and Classical Fields: From Lattice to Field Action

105 minutes — lecturer-first development; student notes deferred until the outline is settled

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Core arc:

$$q_n(t) \text{ with discrete } n \text{ and continuous } t \to \phi(x,t) \text{ with continuous } x, t$$
$$\Downarrow$$
$$\text{local difference operator} \to \text{local differential operator}$$
$$\Downarrow$$
$$\textstyle\sum_n L_n \to \int dx\, \mathcal{L} \to S[\phi] = \int dt\, dx\, \mathcal{L}$$
$$\Downarrow$$
$$\text{field variation} \to \text{bulk equation} + \text{boundary terms}$$
$$\Downarrow$$
$$\text{wave equation, modes, and continuum-to-discrete computation}$$

Protected goals:

- Make explicit that Lecture 8 already treated time as continuous; Lecture 9 makes the spatial label continuous as well.
- Present the real-space limit (dense sites) and mode-space limit (unbounded number of modes) as two views of the same continuum passage.
- Separate locality, linearity, and homogeneity; do not teach "continuum" as synonymous with "linear."
- Separate the continuum/smoothness approximation from geometric or constitutive linearization.
- Derive the Lagrangian density and field Euler–Lagrange equation from the discrete chain rather than quoting them.
- Keep spatial boundary terms visible and physical.
- Close the loop by showing how continuum fields are discretized again for computation: finite differences, spectral expansions, and finite elements.

0–10 min: Lecture 8 already made time continuous

BOARD In Lecture 8 the state was

$$\boxed{q_n(t).}$$

The site label n was discrete, but time t was already continuous.

BOARD Today remove the remaining asymmetry:

$$\boxed{q_n(t) \longrightarrow \phi(x,t).}$$

	Lecture 8	Lecture 9
space	n discrete	x continuous
time	t continuous	t continuous
state	$q_n(t)$	$\phi(x,t)$

SAY A field is not a mysterious new kind of object. At a fixed time, instead of a finite list of generalized coordinates, the configuration is now a whole function $x \mapsto \phi(x,t)$.

BOARD Two equivalent continuum viewpoints:

real space	mode space
sites become dense	number of modes becomes unbounded
$q_n(t) \to \phi(x,t)$	$\displaystyle\sum_{\alpha=1}^{N} a_\alpha(t) u_{\alpha n} \to \sum_{\alpha=1}^{\infty} a_\alpha(t) u_\alpha(x)$

CHECK An infinite number of modes does not require a continuous spectrum. A finite string can have infinitely many discretely labeled modes. A continuous k spectrum needs an additional infinite-domain limit.

10–18 min: Field notation puts space and time on equal footing

BOARD For a field $\phi(x,t)$ define, after saying the full derivatives once,

$$\boxed{\phi_t \equiv \frac{\partial \phi}{\partial t}, \qquad \phi_x \equiv \frac{\partial \phi}{\partial x}, \qquad \phi_{tt} \equiv \frac{\partial^2 \phi}{\partial t^2}, \qquad \phi_{xx} \equiv \frac{\partial^2 \phi}{\partial x^2}.}$$

SAY In particle mechanics time had a special notational role, so dots were natural. In field theory x and t are both independent variables. The compact subscripts help make that symmetry visible. When a derivation would be clearer with the full operator, write it out.

CHECK This is an explicitly declared field-theory shorthand. It should not be confused with compressed notation such as L_q or V_x for derivatives of arbitrary functions.

18–32 min: Take the real-space continuum limit carefully

BOARD Start from the fixed-end chain in the form

$$L = \sum_n \left[\frac{1}{2} m \dot{q}_n^2 - \frac{1}{2} \kappa (q_{n+1} - q_n)^2 \right].$$

Let the spacing be a and write

$$x_n = na, \qquad q_n(t) \to \phi(x,t).$$

BOARD Keep the macroscopic string fixed while refining the lattice:

$$N \to \infty, \qquad a \to 0, \qquad Na \to L.$$

BOARD Scale the discrete parameters so mass and elastic energy remain finite:

$$\boxed{m = \rho a,} \qquad \boxed{\kappa = \frac{\mathcal{T}}{a}.}$$

Here ρ is mass per unit length and $\mathcal{T}$ is the continuum elastic/tension parameter.

BOARD Taylor-expand one nearest-neighbor difference:

$$\phi(x+a,t) - \phi(x,t) = a\phi_x + \frac{a^2}{2}\phi_{xx} + O(a^3).$$

Thus

$$\boxed{q_{n+1} - q_n \simeq a\phi_x.}$$

SAY The approximation is that the field varies slowly over one lattice spacing. Schematically,

$$\boxed{a\,\partial_x \ll 1.}$$

This is a continuum/smoothness assumption. It is not yet a small-amplitude assumption.

32–48 min: Locality, linearity, and homogeneity are different

BOARD Recall the discrete map

$$\boxed{\boldsymbol{e} = A\boldsymbol{q}.}$$

For the simple longitudinal chain, A is fixed and sparse.

BOARD Separate three ideas:

locality	A is sparse: each interaction uses nearby degrees of freedom,
linearity	A does not depend on $\boldsymbol{q}$,
homogeneity	A does not explicitly vary from site to site.

SAY A local theory need not be linear. A linear theory need not be homogeneous.

BOARD Sources of configuration dependence in discrete mechanics include

$$\boxed{A(\boldsymbol{q}), \qquad C(\boldsymbol{q}), \qquad \mathbf{f}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t),}$$

and, in more general coordinates, possibly $M(\boldsymbol{q})$ as well.

SAY $A(\boldsymbol{q})$ is geometric nonlinearity: edge directions or lengths change with configuration. $C(\boldsymbol{q})$ is constitutive nonlinearity. $\mathbf{f}(\boldsymbol{q})$ carries nonlinear forcing or potential structure. All of these can survive the continuum limit.

BOARD In the continuum the local difference operator becomes a local differential operator,

$$\boxed{A \longrightarrow \frac{\partial}{\partial x},}$$

but C and f may remain field-dependent.

BOARD Example of a local nonlinear continuum equation:

$$\rho\phi_{tt} - \frac{\partial}{\partial x}\left[C(\phi, \phi_x, x)\phi_x\right] = f(\phi, \phi_t, x, t).$$

STOP Do not say "continuum means linear." The derivative operator is linear; the theory need not be.

48–58 min: Continuum approximation is not geometric linearization

BOARD For a transverse displacement y_n, the exact length of one segment is

$$\ell_n = \sqrt{a^2 + (y_{n+1} - y_n)^2} = a\sqrt{1 + \left(\frac{y_{n+1} - y_n}{a}\right)^2}.$$

SAY Now geometry itself depends on the configuration. Passing from finite differences to derivatives does not remove that nonlinearity.

BOARD The small-slope approximation is a separate statement:

$$\boxed{\left|\frac{y_{n+1} - y_n}{a}\right| \ll 1 \quad \longrightarrow \quad |\phi_x| \ll 1.}$$

Then

$$\sqrt{1 + \phi_x^2} \simeq 1 + \frac{1}{2}\phi_x^2 + \cdots.$$

BOARD Put the two assumptions side by side:

continuum/smoothness	$a\,\partial_x \ll 1$	variation small over one lattice spacing,
geometric linearization	$\lvert\phi_x\rvert \ll 1$	small slope/strain.

SAY A continuum theory can be fully nonlinear. Linearization is an additional approximation.

58–70 min: From a discrete sum to a Lagrangian density

BOARD Return to the discrete chain:

$$\boxed{L = \sum_n \left[\frac{1}{2} m \dot{q}_n^2 - \frac{1}{2}\kappa (q_{n+1} - q_n)^2\right].}$$

BOARD Now substitute

$$q_n(t) \longrightarrow \phi(x,t), \qquad m = \rho a, \qquad \kappa = \frac{\mathcal{T}}{a}, \qquad q_{n+1} - q_n \simeq a\phi_x.$$

Then one lattice contribution becomes

$$\frac{1}{2}\rho a\, \phi_t^2 - \frac{1}{2}\frac{\mathcal{T}}{a}(a\phi_x)^2 = a\left[\frac{1}{2}\rho\phi_t^2 - \frac{1}{2}\mathcal{T}\phi_x^2\right].$$

BOARD Since

$$\sum_n a \longrightarrow \int dx,$$

we obtain

$$\boxed{L(t) = \int dx\, \mathcal{L},}$$

with

$$\boxed{\mathcal{L} = \frac{1}{2}\rho\phi_t^2 - \frac{1}{2}\mathcal{T}\phi_x^2.}$$

SAY $\mathcal{L}$ is a Lagrangian density. The Lagrangian itself is still the spatial integral.

BOARD Therefore

$$\boxed{S[\phi] = \int dt \int dx\, \mathcal{L} = \int dt\, dx\, \mathcal{L}.}$$

SAY This is the field-theory version of the same action principle used in Lecture 5. The difference is that the configuration variable is now a function of space as well as time.

70–87 min: Vary the field action — bulk plus boundaries

BOARD Take

$$\phi(x,t) \to \phi(x,t) + \epsilon\eta(x,t), \qquad \delta\phi = \epsilon\eta.$$

For a general

$$\mathcal{L} = \mathcal{L}(\phi, \phi_t, \phi_x, x, t),$$

this is a variation of the *field* at fixed spacetime coordinates:

$$\boxed{\delta x = 0, \qquad \delta t = 0.}$$

Thus the full chain rule would contain terms proportional to $(\partial\mathcal{L}/\partial x)\,\delta x$ and $(\partial\mathcal{L}/\partial t)\,\delta t$, but both vanish in this field variation. The explicit derivatives $\partial\mathcal{L}/\partial x$ and $\partial\mathcal{L}/\partial t$ need not themselves vanish. Therefore

$$\delta S = \int dt\, dx \left[\frac{\partial\mathcal{L}}{\partial\phi}\,\delta\phi + \frac{\partial\mathcal{L}}{\partial\phi_t}\,\delta\phi_t + \frac{\partial\mathcal{L}}{\partial\phi_x}\,\delta\phi_x\right].$$

BOARD Use

$$\delta\phi_t = \frac{\partial}{\partial t}(\delta\phi), \qquad \delta\phi_x = \frac{\partial}{\partial x}(\delta\phi).$$

Integrate by parts once in t and once in x.

BOARD The bulk term is

$$\boxed{\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \phi_t}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \phi_x}\right) = 0.}$$

SAY This is the field Euler–Lagrange equation. It is not a new variational principle; it is the same integration-by-parts logic with an additional independent variable.

BOARD Keep the spatial boundary term visible:

$$\boxed{\left[\frac{\partial \mathcal{L}}{\partial \phi_x}\,\delta\phi\right]_{x=x_1}^{x=x_2}.}$$

ASK What physical choices make this vanish?

BOARD Examples:

fixed endpoint	$\delta\phi = 0,$	Dirichlet-type condition,
free/natural endpoint	$\dfrac{\partial \mathcal{L}}{\partial \phi_x} = 0,$	force/flux condition,
periodic	matching values and flux coefficients.	

CHECK For periodic variations, $\delta\phi(x_2, t) = \delta\phi(x_1, t)$. The two spatial terms cancel only if

$$\boxed{\left.\frac{\partial \mathcal{L}}{\partial \phi_x}\right|_{x_2} = \left.\frac{\partial \mathcal{L}}{\partial \phi_x}\right|_{x_1}.}$$

Smooth periodic field and material data ensure this matching; otherwise stationarity imposes it as a natural seam condition, with no added seam action.

SAY The boundary term is physical information, just as it was in the discrete chain and in Lecture 5.

87–95 min: Recover the wave equation

BOARD For

$$\mathcal{L} = \frac{1}{2}\rho\phi_t^2 - \frac{1}{2}\mathcal{T}\phi_x^2,$$

we have

$$\frac{\partial \mathcal{L}}{\partial \phi} = 0, \qquad \frac{\partial \mathcal{L}}{\partial \phi_t} = \rho\phi_t, \qquad \frac{\partial \mathcal{L}}{\partial \phi_x} = -\mathcal{T}\phi_x.$$

Therefore

$$\boxed{\rho\phi_{tt} - \mathcal{T}\phi_{xx} = 0.}$$

Equivalently,

$$\boxed{\phi_{tt} = c^2\phi_{xx}, \qquad c^2 = \frac{\mathcal{T}}{\rho}.}$$

BOARD Connect directly to Lecture 8:

$$\frac{q_{n+1} - 2q_n + q_{n-1}}{a^2} \longrightarrow \phi_{xx}.$$

SAY The wave equation was already present structurally in the discrete chain. The continuum limit exposes it as a differential equation.

95–101 min: The mode-space continuum limit

BOARD For fixed ends try separation

$$\phi(x,t) = X(x)Q(t).$$

The spatial eigenfunctions are

$$\boxed{X_n(x) = \sin\left(\frac{n\pi x}{L}\right), \qquad n = 1, 2, \ldots}$$

and

$$\boxed{\omega_n = c\frac{n\pi}{L}.}$$

SAY Lecture 8 had N eigenvectors. The continuum field has infinitely many eigenfunctions. This is the second route to the same limit.

BOARD

$$\boxed{\phi(x,t) = \sum_{n=1}^{\infty} a_n(t) X_n(x).}$$

101–105 min: Continuum physics, discrete computation

SAY We reached the continuum by refining a discrete physical model. Computers usually make the reverse move: replace a continuum field by finitely many numbers.

BOARD Three common routes:

finite difference	$\phi(x,t) \to \phi_j^n$	values on a grid,
spectral	$\phi(x,t) \to \sum_{\alpha=1}^{N_m} a_\alpha(t) u_\alpha(x)$	finite set of mode amplitudes,
finite element	$\phi(x,t) \to \sum_j q_j(t) N_j(x)$	local basis functions.

SAY The last two are especially useful callbacks. Spectral methods reuse the normal-mode viewpoint. Finite elements recover sparse matrices because local basis functions encode locality.

OPTIONAL A short spectral wave demo would be a useful computational close: truncate

$$\phi(x,t) = \sum_{n=1}^{N_m} a_n(t) \sin(n\pi x/L),$$

evolve each $a_n(t)$ analytically, and reconstruct the field. This gives a clean contrast between finite-difference error in $\Delta x, \Delta t$ and spectral truncation error in the number of retained modes.

BOARD Final conceptual map:

discrete physical model $\to$ continuum field theory $\to$ finite numerical representation

$$q_n(t) \to \phi(x,t) \to \{\phi_j^n\} \text{ or } \{a_\alpha(t)\}.$$

NEXT Use the field viewpoint only as far as needed here. The next lecture changes subject: return to finite-dimensional mechanics and reduce the two-body problem to center-of-mass and relative motion before introducing effective potentials.

If behind: protect the opening discrete-to-continuum bridge, the locality/linearity distinction, the derivation of $\mathcal{L}$, the field Euler–Lagrange equation, and the spatial boundary term. Shorten the transverse-string geometry example, the explicit mode formulas, and the numerical-method survey first.