

Lecture 8 Instructor Outline

Coupled Oscillators as a Discrete Field: Normal Modes and Mode Counting

105 minutes — board/teaching distillation; student notes will expand from this version

Core arc:

mass-spring chain $\rightarrow A\mathbf{q} \rightarrow K = A^T A \rightarrow$ equations of motion $\rightarrow$ eigenvectors
 $\rightarrow$ normal modes $\rightarrow$ mode coordinates $\rightarrow$ ring + symmetry $\rightarrow$ mode counting $\rightarrow$ continuum preview

Protected goals:

- Make clear that the list $(q_1, \dots, q_N)$ is not yet a spatial field: the connectivity encoded by A tells us which degrees of freedom are neighbors and therefore what “local” means.
- Make $A^T A$ emerge from the mechanics of spring extensions and nodal forces, not as a matrix recipe.
- Make a normal mode mean “fixed spatial shape times one scalar oscillation.”
- Show that diagonalizing K is physically a change of generalized coordinates that decouples the motion.
- Relate the fixed-fixed sine modes to the periodic Fourier modes of the ring.
- Distinguish the ring’s discrete site-translation symmetry from the continuous displacement symmetry $q_n \rightarrow q_n + c$.
- Connect the continuous displacement symmetry to a zero restoring force, zero eigenvalue, and zero-frequency mode.
- End with $q_n(t)$ looking naturally like a discrete field that wants to become $\phi(x, t)$.

0–10 min: What makes a collection of coordinates a discrete field?

BOARD Consider N scalar quantities

$$\mathbf{q}(t) = \begin{pmatrix} q_1(t) \\ \vdots \\ q_N(t) \end{pmatrix}.$$

SAY The list of values alone does not yet say that these variables live on a line, a surface, or any spatial object. We need additional structure telling us which sites are neighbors.

BOARD There are two complementary ways to connect a discrete field to a continuum field.

1. Sample or interpolate a continuum object. Think of the nodes as sample points of an underlying field. In one dimension the nodal values can define a piecewise-linear interpolant along line segments. In two dimensions a triangulation gives a piecewise-planar field on each triangle. Delta-function samples or cell averages are other possible representations, but the simplicial piecewise-linear picture makes the geometry visible.

2. Let the interactions define the discrete geometry. The variables may instead be genuine objects, with no continuum assumed in advance. The physical interaction graph first tells us which objects are local neighbors. A chain has one-dimensional neighborhood structure. In higher dimension, identify local subgraphs that are to represent a d -dimensional neighborhood and, when useful, add *virtual geometric edges* that do not represent new physical forces but complete a dimension-preserving simplicial structure. Triangles then supply a two-dimensional simplicial manifold; tetrahedra can play the analogous role in three dimensions.

SAY The physical graph and the simplicial/geometric completion need not be identical. Physical edges encode interactions. Virtual edges and higher simplices may be bookkeeping devices used to expose the local manifold structure.

BOARD Edge differences provide the first local coordinate information:

$$q_j - q_i \quad \text{on an oriented edge } i \rightarrow j.$$

Collections of independent local edge differences act like discrete coordinate directions. Overlapping neighborhoods give the analogue of charts; rules for comparing directions between neighboring charts give a discrete connection or parallel transport. Transport around a closed loop need not return a direction unchanged:

$$\text{nontrivial transport around a loop} \implies \text{discrete curvature.}$$

Metric data may then be added to assign lengths and angles. With a metric-compatible choice of transport, this becomes the discrete analogue of familiar Riemannian geometry.

BOARD

field values $\mathbf{q}$ + local incidence/interactions + simplicial structure
 + transport + metric $\implies$ discrete field geometry

SAY The list $q_1, \dots, q_N$ is therefore not the whole story. The interaction/incidence structure gives the meaning of local, while its simplicial completion can expose effective dimension and topology. Once a metric is supplied and neighboring sites are sufficiently close compared with the scale over which the field varies, a genuinely discrete interacting system and a discretization of a continuum field can have the same long-wavelength description.

ASK If I hand you only the numbers $q_1, \dots, q_N$, can you tell whether they lie on a chain, a ring, or a two-dimensional mesh? What extra information do you need?

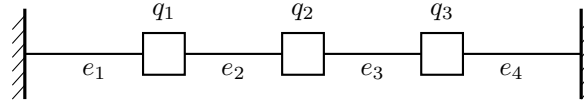
BOARD Preview the destination:

$q_n(t)$ discrete field value at site n .

SAY In this lecture time is already continuous while the spatial label is discrete. The next lecture will make the spatial description continuous as well.

10–22 min: Build the chain from spring extensions

BOARD Start with three moving masses between fixed endpoints:



BOARD Choose the signed spring extensions

$$e_1 = q_1, \quad e_2 = q_2 - q_1, \quad e_3 = q_3 - q_2, \quad e_4 = -q_3.$$

Therefore

$\mathbf{e} = A\mathbf{q},$

 $A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}.$

SAY This is the same node-to-edge idea from the opening lectures. A maps nodal displacements to spring extensions.

BOARD For identical springs of constant κ ,

$$V = \frac{\kappa}{2} \mathbf{e}^T \mathbf{e} = \frac{\kappa}{2} (A\mathbf{q})^T (A\mathbf{q}) = \frac{\kappa}{2} \mathbf{q}^T A^T A \mathbf{q}.$$

Define

$K \equiv A^T A.$

For equal masses,

$$T = \frac{m}{2} \dot{\mathbf{q}}^T \dot{\mathbf{q}},$$

so

$$L = \frac{m}{2} \dot{\mathbf{q}}^T \dot{\mathbf{q}} - \frac{\kappa}{2} \mathbf{q}^T K \mathbf{q}.$$

CHECK With a more general spring matrix C , the earlier structure is $A^T C A$. Here $C = \kappa I$, and κ has simply been factored out.

Likewise, the kinetic energy has a mass-matrix form

$$T = \frac{1}{2} \dot{\mathbf{q}}^T M \dot{\mathbf{q}}.$$

For identical masses,

$$\boxed{M = mI,} \quad \dot{\mathbf{q}}^T M \dot{\mathbf{q}} = \dot{\mathbf{q}}^T (mI) \dot{\mathbf{q}} = m \dot{\mathbf{q}}^T \dot{\mathbf{q}}.$$

So this chain is the particularly transparent case in which both quadratic forms have simple matrix structure: $M = mI$ and, after factoring out κ , $K = A^T A$.

22–32 min: Equations of motion and the discrete second difference

BOARD Euler–Lagrange gives

$$\boxed{m\ddot{\mathbf{q}} = -\kappa K \mathbf{q}.$$

For an interior site,

$$\boxed{m\ddot{q}_n = \kappa(q_{n+1} - 2q_n + q_{n-1}).}$$

ASK What spatial structure do you recognize on the right-hand side?

SAY It is a discrete second difference. Do not call it the wave equation yet. First understand what the discrete mechanics is doing.

BOARD The operator chain has a direct physical meaning:

$$\boxed{\text{nodal displacements } \mathbf{q} \xrightarrow{A} \text{spring extensions } \mathbf{e} \xrightarrow{A^T} \text{net nodal restoring forces.}}$$

Thus $K = A^T A$ is not an arbitrary tridiagonal matrix; it is the composition of two physical maps.

32–42 min: MATLAB at the command line — discover before scripting

SAY Do not open the full Lecture 8 script yet. Build the three-mass problem by hand so the code says nothing we do not already understand.

BOARD MATLAB command-line sequence:

```
A = [ 1  0  0;  
     -1 1  0;  
       0 -1 1;  
       0  0 -1];
```

```
K = A'*A
```

ASK Before pressing return on the last line: what pattern should K have, and why?

BOARD Then:

```
[Q,L] = eig(K);  
diag(L)  
Q  
Q'*Q  
Q'*K*Q  
plot(1:3,Q,'o-')
```

ASK Why are there three eigenvectors? What might the columns of Q mean physically? Why is $Q^T K Q$ diagonal?

SAY The numerical shapes come first. Now derive what they mean.

42–57 min: Normal modes and the fixed–fixed eigenvectors

BOARD Ask for motion that keeps one fixed spatial shape:

$$\boxed{\mathbf{q}(t) = \mathbf{u}_\alpha a_\alpha(t).}$$

Insert into

$$m\ddot{\mathbf{q}} = -\kappa K \mathbf{q} : \\ m\mathbf{u}_\alpha \ddot{a}_\alpha = -\kappa K \mathbf{u}_\alpha a_\alpha.$$

This separates if

$$K \mathbf{u}_\alpha = \lambda_\alpha \mathbf{u}_\alpha.$$

SAY This is the eigenvalue equation. It says that $\mathbf{u}_\alpha$ is a special direction: applying K does not turn it into a new direction. Instead $K \mathbf{u}_\alpha$ is parallel to $\mathbf{u}_\alpha$ and differs only by the scalar factor λ_α .

BOARD

$$K \mathbf{u}_\alpha \parallel \mathbf{u}_\alpha.$$

Then

$$\ddot{a}_\alpha + \frac{\kappa}{m} \lambda_\alpha a_\alpha = 0,$$

so

$$\omega_\alpha^2 = \frac{\kappa}{m} \lambda_\alpha.$$

BOARD Physical definition:

$$\text{normal mode} = \text{fixed spatial shape} \times \text{single-frequency time motion}.$$

CHECK The eigenvector determines the relative motion of all masses. The eigenvalue determines the squared frequency after multiplication by κ/m .

BOARD For the fixed-fixed chain, the boundary conditions suggest standing sine waves. With $n = 1, \dots, N$ labeling sites and $k = 1, \dots, N$ labeling modes, try

$$u_n^{(k)} = \sqrt{\frac{2}{N+1}} \sin\left(\frac{nk\pi}{N+1}\right).$$

The fictitious endpoint values at $n = 0$ and $n = N + 1$ vanish automatically.

SAY This is already a strong plausibility argument: a fixed endpoint wants a node of a standing wave. The only remaining question is whether the discrete second-difference operator sends this shape back into itself.

BOARD Let

$$\theta_k = \frac{k\pi}{N+1}.$$

For an interior site,

$$(K \mathbf{u}^{(k)})_n = 2u_n^{(k)} - u_{n-1}^{(k)} - u_{n+1}^{(k)}.$$

Using

$$\sin((n+1)\theta) + \sin((n-1)\theta) = 2 \cos \theta \sin(n\theta),$$

we get

$$K \mathbf{u}^{(k)} = \lambda_k \mathbf{u}^{(k)}, \quad \lambda_k = 2 \left[1 - \cos\left(\frac{k\pi}{N+1}\right) \right].$$

Equivalently,

$$\lambda_k = 4 \sin^2\left(\frac{k\pi}{2(N+1)}\right).$$

BOARD MATLAB check for a general fixed chain:

```
[Q,L] = eig(K);
[lambda,order] = sort(diag(L));
Q = Q(:,order);
L = diag(lambda);
```

```
n = (1:N)';
k = (1:N)';
```

```

Qexact = sqrt(2/(N+1))*sin(n*k'*pi/(N+1));
Lexact = diag(2*(1-cos(k*pi/(N+1))));

norm(K*Qexact - Qexact*Lexact)
norm(Qexact'*Qexact - eye(N))
norm(L-Lexact)
abs(Q'*Qexact)

```

CHECK The normalization is $\sqrt{2/(N+1)}$; for the $N = 3$ command-line example this is $1/\sqrt{2}$. MATLAB's `eig` eigenvectors may differ from these columns by signs, so `abs(Q'*Qexact)` is the useful comparison.

57–68 min: Eigenvectors are new generalized coordinates

BOARD Collect the orthonormal eigenvectors into

$$Q = (\mathbf{u}_1 \quad \mathbf{u}_2 \quad \cdots \quad \mathbf{u}_N).$$

Because K is real and symmetric,

$$\boxed{Q^T Q = I,} \quad \boxed{Q^T K Q = \Lambda,}$$

with

$$\Lambda = \text{diag}(\lambda_1, \dots, \lambda_N).$$

BOARD Define new generalized coordinates:

$$\boxed{\mathbf{q} = Q\mathbf{a},} \quad \boxed{\mathbf{a} = Q^T \mathbf{q}.$$

Then

$$L = \frac{m}{2} \dot{\mathbf{a}}^T \dot{\mathbf{a}} - \frac{\kappa}{2} \mathbf{a}^T \Lambda \mathbf{a}$$

or

$$\boxed{L = \sum_{\alpha=1}^N \left[\frac{m}{2} \dot{a}_{\alpha}^2 - \frac{\kappa \lambda_{\alpha}}{2} a_{\alpha}^2 \right].}$$

SAY Diagonalizing K is not merely a matrix trick. It is a change of generalized coordinates from site amplitudes to mode amplitudes. In those coordinates the coupled system becomes N independent oscillators.

BOARD Mode counting:

$$\boxed{N \text{ independent coordinates} \iff N \text{ independent normal modes.}}$$

OPTIONAL Do not develop the full $M\ddot{\mathbf{q}} + K\mathbf{q} = 0$ generalized-eigenvalue theory here. That belongs in the later small-oscillation unit. The equal-mass chain lets the structure remain visible.

68–80 min: Change the topology — close the chain into a ring

SAY Now remove the fixed endpoints and connect the last site to the first. Every spring still depends only on differences of neighboring displacements.

BOARD For the ring,

$$A\mathbf{1} = \mathbf{0} \implies K\mathbf{1} = A^T A\mathbf{1} = \mathbf{0}.$$

Therefore the uniform vector is an eigenvector with

$$\boxed{\lambda_0 = 0,} \quad \boxed{\omega_0 = 0.}$$

ASK Why should this zero mode have existed before we ever diagonalized a matrix?

BOARD Because the transformation

$$q_n \rightarrow q_n + c \quad \text{for every } n$$

leaves every spring extension $q_{n+1} - q_n$ unchanged. Therefore V is unchanged.

SAY This is a continuous translation in configuration space, exactly of the kind discussed in Lecture 7. Its infinitesimal generator is the uniform vector $\mathbf{1}$.

BOARD Noether quantity:

$$J = \sum_n p_n = m \sum_n \dot{q}_n = \text{constant}.$$

And the same symmetry appears in the stiffness matrix as

$$\text{continuous displacement symmetry} \iff \text{zero restoring force} \iff \text{zero eigenvalue} \iff \text{zero mode}.$$

STOP Do not confuse this with the ring's discrete site translation $n \rightarrow n+1$. The site relabeling is a discrete spatial symmetry; $q_n \rightarrow q_n + c$ is a continuous configuration-space symmetry and is the one associated with the Noether charge above.

80–92 min: Fourier modes and mode counting on the ring

SAY The fixed chain already gave standing sine modes. Closing the chain changes only the global boundary condition; the local second-difference operator is the same. On the ring, translation symmetry makes complex Fourier waves the natural basis.

BOARD Compare the allowed wave numbers:

$$\begin{array}{ll} \text{fixed-fixed:} & k_j a = \frac{j\pi}{N+1}, \quad j = 1, \dots, N, \\ \text{periodic ring:} & k_r a = \frac{2\pi r}{N}, \quad r = 0, \dots, N-1. \end{array}$$

In both cases the same local operator has eigenvalue

$$\lambda(k) = 2 - 2\cos(ka) = 4\sin^2(ka/2),$$

but the boundary condition selects the allowed values of k and the convenient basis.

SAY Because every site has the same local surroundings, the ring suggests a spatial pattern that reproduces itself under a one-site shift.

BOARD Try

$$u_n = e^{ikna}.$$

Periodic closure requires

$$u_{n+N} = u_n,$$

so

$$e^{ikNa} = 1.$$

Therefore

$$k_r = \frac{2\pi r}{Na}, \quad r = 0, 1, \dots, N-1.$$

There are exactly N allowed wave numbers.

BOARD Apply K to the plane-wave pattern:

$$\lambda(k) = 2 - e^{ika} - e^{-ika} = 2 - 2\cos(ka) = 4\sin^2\left(\frac{ka}{2}\right).$$

Thus

$$\omega^2(k) = \frac{4\kappa}{m} \sin^2\left(\frac{ka}{2}\right).$$

SAY This is our first dispersion relation, but do not turn today into a dispersion-relation lecture. The important point is that spatial symmetry organizes the normal modes and periodic closure counts them.

CHECK In complex notation, $+k$ and $-k$ appear as a conjugate pair. For a real symmetric K , MATLAB may return real sine/cosine combinations spanning the same degenerate subspace. Different basis, same mode space.

92–101 min: Now open the complete MATLAB script

SAY At this point the script should look like a transcription of the mechanics rather than a black box.

BOARD Walk through only the structural lines first:

```
K = A'*A;
[Q,LambdaMatrix] = eig(K);
...
q = q + ampList(j)*Q(:,modeList(j));
...
v = v - (kappa/m)*(K*q)*dt;
q = q + v*dt;
```

SAY The time update is the same velocity-first symplectic-Euler structure used earlier in the course: update the velocity from the force, then update the configuration from the new velocity.

BOARD The revealing projection is

```
a = Q'*qs;
```

which converts the stored site-coordinate motion into normal-mode coordinates.

ASK What should a complicated-looking motion in $q_n(t)$ look like after projection into $a_\alpha(t)$?

SAY Run the fixed chain first. Then, if time permits, change one line to the periodic ring and use the printed zero-mode check to reconnect the computation to the symmetry argument.

101–105 min: The discrete field wants to become continuous

BOARD Return to the interior equation

$$m\ddot{q}_n = \kappa(q_{n+1} - 2q_n + q_{n-1}).$$

If the site spacing is a , then for slowly varying motion,

$$\boxed{\frac{q_{n+1} - 2q_n + q_{n-1}}{a^2} \longrightarrow \frac{\partial^2 \phi}{\partial x^2}.$$

Thus

$$\boxed{q_n(t) \longrightarrow \phi(x, t).$$

SAY Do not yet derive the continuum field equation. The question for the next lecture is deeper: what happens to the Lagrangian and the action when the discrete spatial index itself becomes continuous?

BOARD Final conceptual map:

$$\begin{array}{c} \mathbf{e} = A\mathbf{q} \\ \Downarrow \\ V = \frac{\kappa}{2} \mathbf{q}^T A^T A \mathbf{q} \\ \Downarrow \\ m\ddot{\mathbf{q}} = -\kappa K \mathbf{q} \\ \Downarrow \\ K \mathbf{u}_\alpha = \lambda_\alpha \mathbf{u}_\alpha \\ \Downarrow \\ \omega_\alpha^2 = (\kappa/m) \lambda_\alpha \\ \Downarrow \\ \mathbf{q} = Q\mathbf{a} \\ \Downarrow \\ N \text{ coupled coordinates} \rightarrow N \text{ independent modes} \\ \Downarrow \\ q_n(t) \rightarrow \phi(x, t) \end{array}$$

NEXT Take the continuum limit seriously: replace the discrete site label by a spatial coordinate, replace the sum in the action by a spatial integral, and derive the field Euler–Lagrange equation with its boundary terms visible.

If behind: shorten the explicit three-mass matrix arithmetic, the detailed MATLAB check of the fixed–fixed formulas, the complex Fourier-mode discussion, and the full-script walk-through first. Do *not* cut the two interpretations of a discrete field, the construction $A \rightarrow A^T A$, the normal-mode ansatz, the fixed-versus-periodic boundary comparison, the ring zero-mode/symmetry connection, or the continuum preview.