

Lecture 7 Instructor Outline

Constraint Forces, Continuous Symmetry, and Noether's Theorem

105 minutes — student notes are canonical; this is the board/teaching distillation

Core arc:

constraint surface $\rightarrow$ reaction direction $\rightarrow$ multiplier $\rightarrow$ augmented L
 $\rightarrow$ cyclic coordinate $\rightarrow$ hidden continuous symmetry $\rightarrow$ action boundary term
 $\rightarrow$ Noether charge $\rightarrow$ translation/rotation/time translation

Protected goals:

- Make the multiplier term emerge from the physical reaction force, not as an unexplained recipe.
- Keep actual time evolution distinct from fixed-time virtual and symmetry variations.
- Make the action derivation of Noether primary: equations of motion remove the bulk term; symmetry makes the endpoint values of the Noether quantity agree.
- State the logic carefully: continuous symmetry *plus* the equations of motion/stationary action gives conservation.

0–5 min: Return to the Lecture 6 pendulum

BOARD Recall the reduced-coordinate description:

$$x = \ell_0 \sin \theta, \quad y = -\ell_0 \cos \theta,$$

$$L = \frac{1}{2} m \ell_0^2 \dot{\theta}^2 + mg \ell_0 \cos \theta,$$

so

$$\frac{d}{dt} (m \ell_0^2 \dot{\theta}) = -mg \ell_0 \sin \theta.$$

ASK Where did the tension go?

SAY Lecture 6 built the constraint into the coordinate choice. We kept the tangent motion and eliminated the radial reaction from the reduced equation. Today we first do the complementary thing: keep the constraint explicitly so that we can recover the force that enforces it.

5–12 min: Constraint surface, generalized coordinates, and time

BOARD Let the coordinates we retain be

$$\mathbf{q} = (q_1, \dots, q_n)^T,$$

not necessarily yet reduced to an independent set. One holonomic constraint is

$$F(\mathbf{q}, t) = 0.$$

SAY Cartesian coordinates are one possible choice of $\mathbf{q}$, but not the only one. If instead we parameterize the allowed surface with independent generalized coordinates so that the constraint is satisfied identically, then the constraint force is projected out exactly as in Lecture 6.

BOARD Along the actual motion,

$$\dot{F} = \frac{\partial F}{\partial \mathbf{q}} \cdot \dot{\mathbf{q}} + \frac{\partial F}{\partial t} = 0.$$

For a virtual displacement,

$$\delta t = 0,$$

$$\delta F = \frac{\partial F}{\partial \mathbf{q}} \cdot \delta \mathbf{q} = 0.$$

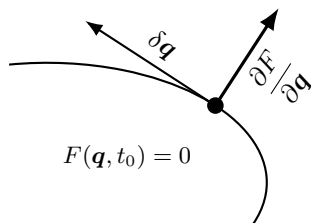


Figure 1: At fixed time, allowed virtual displacements are tangent to the instantaneous constraint surface and $\partial F / \partial \mathbf{q}$ is normal.

SAY The normal shown in the diagram is the derivative with respect to the configuration coordinates. It does not contain $\partial F / \partial t$. For a moving constraint the actual velocity need not be tangent to the instantaneous surface because

$$\frac{\partial F}{\partial \mathbf{q}} \cdot \dot{\mathbf{q}} = -\frac{\partial F}{\partial t},$$

but a virtual displacement is always taken at the same time and is tangent.

12–18 min: Ideal reaction force and the choice of constraint function

BOARD Lecture 6 allowed a general generalized force on the right-hand side of Lagrange's equation. Here the generalized force of interest is the reaction force:

$$\mathbf{R} \equiv \mathbf{Q}^{(R)}.$$

For an ideal constraint,

$$\mathbf{R} \cdot \delta \mathbf{q} = 0$$

for every allowed $\delta \mathbf{q}$. Since those variations are tangent,

$$\mathbf{R} = \lambda \frac{\partial F}{\partial \mathbf{q}}.$$

SAY $\mathbf{R}$ is a generalized reaction-force vector. Its components are conjugate to the chosen coordinates; they need not all have ordinary force units. The multiplier λ gives the magnitude only relative to the particular defining function F .

BOARD The physical object is the constraint surface, not the particular formula used to describe it. For the pendulum constraint $\ell = \ell_0$:

$F(\ell)$	$\frac{\partial F}{\partial \ell}$ at ℓ_0	multiplier use
$\ell - \ell_0$	1	good
$(\ell - \ell_0)^2$	0	bad: no normal direction there
$ \ell - \ell_0 $	undefined	bad: not differentiable
$\ell^2 - \ell_0^2$	$2\ell_0$	good locally if $\ell_0 > 0$

CHECK A useful defining function is differentiable and regular on the constraint surface:

$$\frac{\partial F}{\partial \mathbf{q}} \neq 0 \quad \text{on } F = 0.$$

If $\tilde{F} = g(F)$ with $g(0) = 0$ and $g'(0) \neq 0$, then on the constraint

$$\frac{\partial \tilde{F}}{\partial \mathbf{q}} = g'(0) \frac{\partial F}{\partial \mathbf{q}},$$

so the multiplier rescales while the physical reaction does not:

$$\mathbf{R} = \lambda \frac{\partial F}{\partial \mathbf{q}} = \tilde{\lambda} \frac{\partial \tilde{F}}{\partial \mathbf{q}}.$$

18–23 min: Discover the augmented Lagrangian from the force equation

BOARD The forced Euler–Lagrange equation is

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\mathbf{q}}} \right) - \frac{\partial L}{\partial \mathbf{q}} = \lambda \frac{\partial F}{\partial \mathbf{q}}.$$

Move the reaction term to the left:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\mathbf{q}}} \right) - \frac{\partial L}{\partial \mathbf{q}} - \lambda \frac{\partial F}{\partial \mathbf{q}} = 0.$$

Now notice

$$-\frac{\partial L}{\partial \mathbf{q}} - \lambda \frac{\partial F}{\partial \mathbf{q}} = -\frac{\partial}{\partial \mathbf{q}}(L + \lambda F),$$

where $\lambda(t)$ is an independent multiplier variable. Since a holonomic constraint has no $\dot{\mathbf{q}}$ dependence,

$$\frac{\partial}{\partial \dot{\mathbf{q}}}(L + \lambda F) = \frac{\partial L}{\partial \dot{\mathbf{q}}}.$$

Therefore

$$\frac{d}{dt} \left[\frac{\partial}{\partial \dot{\mathbf{q}}}(L + \lambda F) \right] - \frac{\partial}{\partial \mathbf{q}}(L + \lambda F) = 0.$$

Only now define

$$L_\lambda \equiv L + \lambda F.$$

Variation with respect to λ returns

$$F(\mathbf{q}, t) = 0.$$

SAY The term λF is not a trick for inventing the constraint force. Geometry and virtual work told us the form of the reaction first. The augmented Lagrangian packages that already-understood reaction into the variational machinery.

23–35 min: Pendulum — mirror the whole construction

BOARD Instead of building the rod length in from the start, retain

$$\mathbf{q} = \begin{pmatrix} \ell \\ \theta \end{pmatrix}, \quad F(\ell, \theta) = \ell - \ell_0 = 0.$$

Then

$$\frac{\partial F}{\partial \mathbf{q}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

so

$$\mathbf{R} = \lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

SAY Before doing any dynamics we know the constraint reaction appears only in the radial equation and not in the angular equation. In this normalization, λ itself is the signed radial generalized reaction force.

BOARD With

$$x = \ell \sin \theta, \quad y = -\ell \cos \theta,$$

we have

$$T = \frac{1}{2}m \left(\dot{\ell}^2 + \ell^2 \dot{\theta}^2 \right), \quad V = -mg\ell \cos \theta,$$

so

$$L = \frac{1}{2}m \left(\dot{\ell}^2 + \ell^2 \dot{\theta}^2 \right) + mg\ell \cos \theta.$$

The augmented Lagrangian is therefore

$$L_\lambda = L + \lambda(\ell - \ell_0).$$

BOARD Angular equation:

$$\frac{d}{dt} (m\ell^2 \dot{\theta}) = -mg\ell \sin \theta.$$

On the constraint $\ell = \ell_0$,

$$\ddot{\theta} + \frac{g}{\ell_0} \sin \theta = 0,$$

exactly the Lecture 6 equation.

BOARD Radial equation:

$$m\ddot{\ell} - m\ell\dot{\theta}^2 - mg \cos \theta = \lambda.$$

On the constraint,

$$\ell = \ell_0, \quad \dot{\ell} = \ddot{\ell} = 0,$$

so

$$\lambda = -m\ell_0\dot{\theta}^2 - mg \cos \theta.$$

Positive ℓ points outward. Therefore the inward radial reaction is

$$R_{\text{inward}} = -\lambda = m\ell_0\dot{\theta}^2 + mg \cos \theta.$$

For a taut string this is the tension.

CHECK Put the two descriptions side by side:

$\ell = \ell_0$ built into the coordinates	$\longrightarrow$	one coordinate θ , reaction disappears,
ℓ retained and $F = 0$	$\longrightarrow$	two coordinates + multiplier, reaction recovered.

OPTIONAL The full student notes may also show the equivalent Cartesian calculation with x, y and $x^2 + y^2 - \ell_0^2 = 0$. Do not spend board time repeating it.

35–42 min: Cyclic coordinate = translation symmetry in coordinate space

SAY From here on, return to independent generalized coordinates $\mathbf{q}$ unless stated otherwise.

BOARD

$$p_j \equiv \frac{\partial L}{\partial \dot{q}_j}, \quad \dot{p}_j = \frac{\partial L}{\partial q_j}.$$

Lecture 6 ended with

$$\frac{\partial L}{\partial q_k} = 0 \implies p_k = \text{constant.}$$

ASK What does $\partial L / \partial q_k = 0$ mean geometrically?

BOARD

$$q_k \rightarrow q_k + \epsilon$$

leaves L unchanged. Therefore a cyclic coordinate already describes a continuous symmetry.

SAY Here “translation” means translation in a generalized coordinate. q_k might be Cartesian position, an angle, or something less familiar.

42–52 min: A conserved quantity can be hidden by the coordinates

BOARD Two particles in one dimension:

$$L = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - V(x_1 - x_2).$$

Neither x_1 nor x_2 is cyclic separately.

BOARD Common translation:

$$x_1 \rightarrow x_1 + \epsilon, \quad x_2 \rightarrow x_2 + \epsilon.$$

The separation and velocities are unchanged, so L is unchanged.

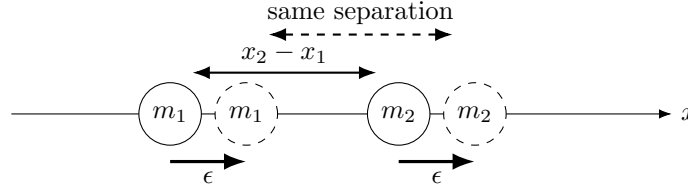


Figure 2: A common translation changes absolute positions but leaves the interaction coordinate unchanged.

ASK We know that if L does not depend on one q_n , then one momentum is conserved. Here neither original coordinate is absent from L , yet there is still a conservation law. Is there something deeper than “cyclic coordinate”?

SAY Yes. The more general structure is symmetry.

OPTIONAL Make the symmetry manifest with

$$X = \frac{m_1x_1 + m_2x_2}{M}, \quad r = x_1 - x_2,$$

so

$$L = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}\mu\dot{r}^2 - V(r),$$

and X is cyclic.

52–58 min: What kind of symmetry gives a Noether charge?

SAY A symmetry is a change in the description that leaves the relevant physical structure unchanged. Rotating a circle leaves the circle unchanged. A square also has rotational symmetry, but only for special finite angles. The difference matters.

BOARD

continuous symmetry = a symmetry connected to the identity by arbitrarily small changes.

A circle has continuous rotational symmetry. A square or equilateral triangle has only discrete rotational symmetry.

SAY Noether’s theorem in the form we need uses the continuous case because an infinitesimal change can be generated at every point along the trajectory. Symmetry alone is not yet a conservation law: the conservation statement will use both the symmetry and the equations of motion.

BOARD At fixed time, write a small transformation as

$$q'_n = q_n + \epsilon h_n(\mathbf{q}, t),$$

$$\delta_s q_n = \epsilon h_n(\mathbf{q}, t),$$

$$\delta_s t = 0.$$

The functions h_n give the components of the infinitesimal symmetry displacement in configuration space.

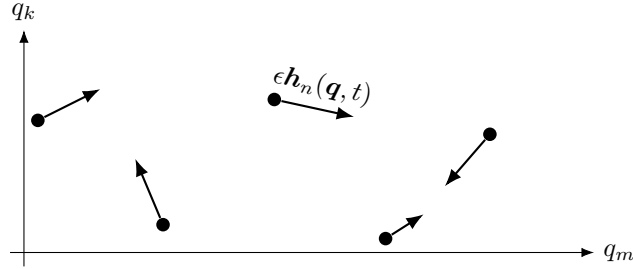
SAY Here n labels a scalar generalized-coordinate component. For N particles in d dimensions, the same dN -component objects can either be flattened,

$$\mathbf{q} = (q_1, \dots, q_{dN})^T, \quad \mathbf{h} = (h_1, \dots, h_{dN})^T,$$

or regrouped into particle blocks,

$$\mathbf{q} = \begin{pmatrix} q_1 \\ \vdots \\ q_N \end{pmatrix}, \quad \mathbf{h} = \begin{pmatrix} h_1 \\ \vdots \\ h_N \end{pmatrix}, \quad \mathbf{q}_i, \mathbf{h}_i \in \mathbb{R}^d.$$

Thus h_n is a scalar component in the generic derivation, while $\mathbf{h}_i$ later denotes the d -vector block for particle i . They are two organizations of the same configuration-space vector field.



An infinitesimal transformation field on configuration space.

Figure 3: The generator $\mathbf{h}(\mathbf{q}, t)$ assigns an infinitesimal displacement direction to each configuration. The drawing shows its projection onto the (q_k, q_m) plane.

58–71 min: Action viewpoint — the boundary term becomes the conserved quantity

SAY Return to the path picture from the variational lectures, but make one crucial change. Previously the trial variation vanished at the endpoints. A symmetry transformation moves the entire trajectory, including its endpoints.

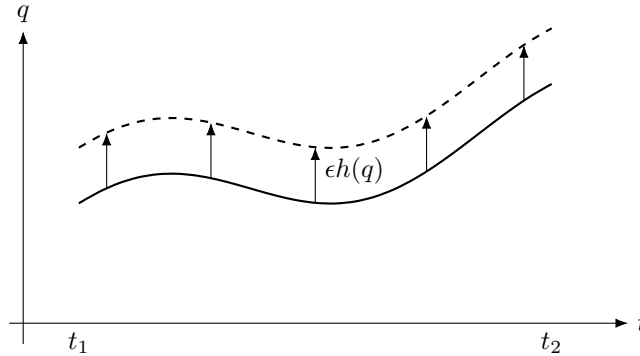


Figure 4: A fixed-time symmetry transformation moves the entire trajectory, including the endpoint configurations. Note: $\delta_s t = 0$: the arrows are vertical in q and do not translate time.

BOARD Before integration by parts,

$$\delta_s \mathcal{S} = \int_{t_1}^{t_2} \sum_n \left[\frac{\partial L}{\partial q_n} \delta_s q_n + \frac{\partial L}{\partial \dot{q}_n} \delta_s \dot{q}_n \right] dt.$$

Integrate the second term by parts:

$$\delta_s \mathcal{S} = \int_{t_1}^{t_2} \sum_n \left[\frac{\partial L}{\partial q_n} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_n} \right) \right] \delta_s q_n dt + \left[\sum_n p_n \delta_s q_n \right]_{t_1}^{t_2}.$$

STOP In the old Hamilton-principle derivation we imposed $\delta q_n(t_1) = \delta q_n(t_2) = 0$ and killed this boundary term. Do not do that here. The symmetry moves the endpoints.

BOARD On a physical trajectory, the Euler–Lagrange equations make the bulk integral zero. Thus

$$\delta_s \mathcal{S} = \left[\sum_n p_n \delta_s q_n \right]_{t_1}^{t_2}.$$

Now insert $\delta_s q_n = \epsilon h_n$:

$$\delta_s \mathcal{S} = \epsilon \left[\sum_n p_n h_n \right]_{t_1}^{t_2}.$$

If the transformation is an exact symmetry of the action,

$$\delta_s \mathcal{S} = 0,$$

so define

$$J(t) \equiv \sum_n p_n(t) h_n(\mathbf{q}(t), t)$$

and obtain

$$J(t_2) - J(t_1) = 0.$$

Because t_1 and t_2 are arbitrary,

$$J = \text{constant}.$$

CHECK State the logic explicitly:

$$\underbrace{\text{equations of motion}}_{\text{bulk term}=0} + \underbrace{\text{continuous action symmetry}}_{\delta_s \mathcal{S}=0} \implies \underbrace{J(t_2) = J(t_1)}_{\text{conservation}}.$$

71–78 min: First checks — cyclic coordinate and spatial translation

BOARD Cyclic coordinate:

$$q_k \rightarrow q_k + \epsilon \implies h_k = 1, \quad h_{n \neq k} = 0.$$

Therefore

$$J = p_k.$$

The old cyclic-coordinate result is the simplest Noether example.

BOARD Spatial translation of every particle by a constant direction $\mathbf{a}$:

$$\mathbf{r}_i \rightarrow \mathbf{r}_i + \epsilon \mathbf{a}, \quad \mathbf{h}_i = \mathbf{a}.$$

Then

$$J = \sum_i \mathbf{p}_i \cdot \mathbf{a} = \mathbf{a} \cdot \sum_i \mathbf{p}_i.$$

If the action is invariant under translations in the direction $\mathbf{a}$, then

$$\mathbf{a} \cdot \mathbf{P} = \text{constant}, \quad \mathbf{P} \equiv \sum_i \mathbf{p}_i.$$

If all spatial translations are symmetries,

$$\mathbf{P} = \text{constant}.$$

SAY For the two-particle example, the common translation leaves $x_1 - x_2$ unchanged, so the interaction and the action are unchanged. The equations of motion turn that symmetry into conservation of total momentum.

78–90 min: Rotation — build the infinitesimal generator before the charge

BOARD Start in two dimensions:

$$\mathbf{r}(\theta) = \begin{pmatrix} r \cos \theta \\ r \sin \theta \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

For a small rotation $\theta \rightarrow \theta + \epsilon$,

$$\mathbf{r}(\theta + \epsilon) - \mathbf{r}(\theta) = \epsilon \frac{d\mathbf{r}}{d\theta} + O(\epsilon^2).$$

Thus, to first order,

$$\delta_s \mathbf{r} = \epsilon \frac{d\mathbf{r}}{d\theta} = \epsilon \begin{pmatrix} -r \sin \theta \\ r \cos \theta \end{pmatrix} = \epsilon \begin{pmatrix} -y \\ x \end{pmatrix}.$$

Therefore

$$\boxed{\delta_s x = -\epsilon y}, \quad \boxed{\delta_s y = \epsilon x}.$$

SAY The infinitesimal rotational displacement is tangent to the circular orbit.

BOARD Geometric-algebra form with $I = \hat{x}\hat{y}$:

$$\mathbf{r}I = (x\hat{x} + y\hat{y})\hat{x}\hat{y} = x\hat{y} - y\hat{x},$$

so

$$\boxed{\delta_s \mathbf{r} = \epsilon \mathbf{r} I}.$$

In ordinary three-vector language,

$$\boxed{\delta_s \mathbf{r} = \epsilon \hat{z} \times \mathbf{r}}.$$

For rotation about a general axis $\hat{n}$,

$$\boxed{\mathbf{h}_i = \hat{n} \times \mathbf{r}_i}.$$

BOARD The Noether quantity is

$$J = \sum_i \mathbf{p}_i \cdot (\hat{n} \times \mathbf{r}_i) = \hat{n} \cdot \sum_i (\mathbf{r}_i \times \mathbf{p}_i).$$

Define

$$\boxed{\mathbf{L} \equiv \sum_i \mathbf{r}_i \times \mathbf{p}_i}.$$

Rotational symmetry about $\hat{n}$ gives

$$\boxed{\hat{n} \cdot \mathbf{L} = \text{constant}}.$$

Full rotational symmetry gives

$$\boxed{\mathbf{L} = \text{constant}}.$$

SAY Again, rotational symmetry by itself is not the conservation law. Rotational invariance of the action together with the equations of motion gives conservation of the corresponding component of angular momentum.

90–99 min: Local differential form — the same theorem point by point

SAY Now compress the global action argument into a local differential statement. We still hold time fixed under the symmetry transformation: $\delta_s t = 0$. The generator $h_n(\mathbf{q}, t)$ may nevertheless depend explicitly on t .

BOARD

$$\delta_s q_n = \epsilon h_n, \quad \delta_s \dot{q}_n = \epsilon \dot{h}_n.$$

The first-order change in L is

$$\delta_s L = \sum_n \left(\frac{\partial L}{\partial q_n} \delta_s q_n + \frac{\partial L}{\partial \dot{q}_n} \delta_s \dot{q}_n \right).$$

Therefore

$$\frac{\delta_s L}{\epsilon} = \sum_n \left(\frac{\partial L}{\partial q_n} h_n + p_n \dot{h}_n \right).$$

On a physical trajectory,

$$\frac{\partial L}{\partial q_n} = \dot{p}_n,$$

so

$$\boxed{\frac{\delta_s L}{\epsilon} = \sum_n (\dot{p}_n h_n + p_n \dot{h}_n) = \frac{d}{dt} \left(\sum_n p_n h_n \right) = \dot{J}.}$$

Thus pointwise L -invariance,

$$\delta_s L = 0,$$

gives

$$\boxed{\dot{J} = 0.}$$

CHECK This is the local version of the action/boundary argument. The product rule is the mechanism.

OPTIONAL If

$$\delta_s L = \epsilon \frac{dB}{dt},$$

then

$$\boxed{J - B = \text{constant}.}$$

This is the quasi-invariant extension; the action can change only by an endpoint term.

99–105 min: Time translation and energy — separate the change of the clock

SAY Everything so far transformed configuration variables while holding the time coordinate fixed:

$$\boxed{\delta_s t = 0.}$$

A time translation changes the independent variable itself, so treat it separately rather than forcing it into the fixed-time derivation.

BOARD Define

$$\boxed{E_L \equiv \sum_j p_j \dot{q}_j - L(\mathbf{q}, \dot{\mathbf{q}}, t),}$$

with

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{q}}}.$$

Taking a total time derivative,

$$\begin{aligned} \dot{E}_L &= \sum_j (\dot{p}_j \dot{q}_j + p_j \ddot{q}_j) - \frac{dL(\mathbf{q}, \dot{\mathbf{q}}, t)}{dt} \\ &= \sum_j (\dot{p}_j \dot{q}_j + p_j \ddot{q}_j) - \left[\sum_j \frac{\partial L}{\partial q_j} \dot{q}_j + \sum_j \frac{\partial L}{\partial \dot{q}_j} \ddot{q}_j + \frac{\partial L}{\partial t} \right] \\ &= \sum_j (\dot{p}_j \dot{q}_j + p_j \ddot{q}_j) - \sum_j \dot{p}_j \dot{q}_j - \sum_j p_j \ddot{q}_j - \frac{\partial L}{\partial t}. \end{aligned}$$

Where we used Euler–Lagrange,

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\mathbf{q}}} \right) = \dot{\mathbf{p}} = \frac{\partial L}{\partial \mathbf{q}}.$$

Then,

$$\dot{E}_L = -\frac{\partial L}{\partial t}.$$

Hence, if

$$\frac{\partial L}{\partial t} = 0,$$

then

$$E_L = \text{constant}.$$

SAY No explicit t means no preferred time origin. For ordinary $L = T - V$ with T quadratic in velocities and velocity-independent V ,

$$E_L = T + V.$$

BOARD Final conceptual map:

translation symmetry + dynamics	$\longrightarrow$	$\mathbf{P}$ conserved,
rotation symmetry + dynamics	$\longrightarrow$	$\mathbf{L}$ conserved,
time-translation symmetry + dynamics	$\longrightarrow$	E_L conserved.

SAY The table is not the theorem. The theorem is the mechanism: a continuous symmetry of the action, together with the equations of motion, turns a boundary quantity into a conserved quantity.

NEXT Carry the symmetry viewpoint into a many-degree-of-freedom system: a coupled oscillator chain, where spatial structure, normal modes, and a zero mode will connect symmetry to collective motion and prepare the transition to fields.

If behind: shorten the regular-defining-function examples, the optional center-of-mass rewrite, the GA rotation aside, and the quasi-invariant extension first. If necessary, make time translation a closing preview. Do *not* cut the physical origin of $\mathbf{R} = \lambda \partial F / \partial \mathbf{q}$, the pendulum mirror calculation, the hidden-translation motivation, the action boundary-term derivation, or the translation/rotation Noether checks.