

Lecture 6 Instructor Outline

Generalized Coordinates, Constraints, and Virtual Work

105 minutes — student notes are canonical; this is the board/teaching distillation

Core arc:

constraints $\rightarrow$ generalized coordinates $\rightarrow$ virtual displacements $\rightarrow$ generalized forces $\rightarrow$ Lagrange

Protected goal: make clear why coordinates adapted to constraints remove redundant variables and, for ideal constraints, eliminate unknown constraint forces from the equations of motion. Protect the distinction $d\mathbf{r}$ versus $\delta\mathbf{r}$ and the d'Alembert projection. Compress terminology and mass-matrix detail first if time is short.

0–10 min: Why change coordinates?

BOARD Recall Lecture 5:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0$$

SAY We used q_j deliberately. The coordinates describing a system need not be Cartesian, and constraints often tell us that Cartesian coordinates contain redundancy.

BOARD Particle on a circle:

$$x^2 + y^2 = R^2, \quad x = R \cos \theta, \quad y = R \sin \theta.$$

Two Cartesian numbers, one independent coordinate:

$$q = \theta.$$

ASK What has the constraint done? *It has not added a degree of freedom; it has removed one.*

SAY x, y are not wrong. They are redundant. Generalized coordinates label the independent directions in configuration space.

10–23 min: Constraints, degrees of freedom, coordinate map

BOARD For N particles in 3D, begin with $3N$ Cartesian coordinates. For m independent holonomic constraints,

$$f_\alpha(\mathbf{r}_1, \dots, \mathbf{r}_N, t) = 0, \quad \alpha = 1, \dots, m$$

locally

$$s = 3N - m$$

independent generalized coordinates remain.

BOARD

$$\mathbf{r}_i = \mathbf{r}_i(q_1, \dots, q_s, t) = \mathbf{r}_i(\mathbf{q}, t).$$

SAY The constraint surface is the allowed configuration space; the q_j are coordinates on it. A generalized coordinate need not have units of length and is not unique.

OPTIONAL Vocabulary only if useful:

- scleronomous: no explicit t in the constraint;
- rheonomous: explicit t dependence;
- nonholonomic: not reducible to equations $f_\alpha(\mathbf{r}, t) = 0$.

Do not let terminology consume the lecture.

23–34 min: Velocities and kinetic energy follow from the map

BOARD Differentiate the coordinate map:

$$\dot{\mathbf{r}}_i = \sum_j \frac{\partial \mathbf{r}_i}{\partial q_j} \dot{q}_j + \frac{\partial \mathbf{r}_i}{\partial t}.$$

SAY This is just the total derivative, but it is the mechanism that converts geometry into dynamics.

BOARD

$$\mathbf{q} \rightarrow \mathbf{r}_i(\mathbf{q}, t) \rightarrow \dot{\mathbf{r}}_i(\mathbf{q}, \dot{\mathbf{q}}, t) \rightarrow T(\mathbf{q}, \dot{\mathbf{q}}, t).$$

BOARD Circle example:

$$\mathbf{r} = R \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \quad T = \frac{1}{2} m R^2 \dot{\theta}^2.$$

SAY Once the coordinate is an angle, kinetic energy is not $\frac{1}{2} m \dot{q}^2$. Geometry appears in the coefficient.

OPTIONAL Mass-matrix glimpse for a time-independent map:

$$T = \frac{1}{2} \dot{\mathbf{q}}^T M(\mathbf{q}) \dot{\mathbf{q}}, \quad M_{jk} = \sum_i m_i \frac{\partial \mathbf{r}_i}{\partial q_j} \cdot \frac{\partial \mathbf{r}_i}{\partial q_k}.$$

34–49 min: Actual displacement versus virtual displacement

BOARD

$$d\mathbf{r}_i = \sum_j \frac{\partial \mathbf{r}_i}{\partial q_j} dq_j + \frac{\partial \mathbf{r}_i}{\partial t} dt.$$

SAY This compares nearby points along an actual motion as time advances.

BOARD A virtual displacement compares nearby allowed configurations at the *same time*:

$$\delta t = 0, \quad \delta \mathbf{r}_i = \sum_j \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j.$$

Hence, in general,

$$d\mathbf{r}_i \neq \delta \mathbf{r}_i.$$

STOP Do not rush this distinction. It is the conceptual hinge of the lecture.

BOARD Constraint geometry:

$$f(\mathbf{r}, t) = 0 \implies \delta f = \nabla f \cdot \delta \mathbf{r} = 0.$$

Therefore $\delta \mathbf{r}$ is tangent to the constraint surface.

BOARD Circle:

$$x^2 + y^2 = R^2 \implies x \delta x + y \delta y = 0.$$

ASK Which direction is forbidden? *The normal/radial direction.*

49–61 min: Ideal constraints and virtual work

BOARD For a smooth frictionless constraint, reaction force $\mathbf{R}$ is normal while $\delta \mathbf{r}$ is tangent:

$$\mathbf{R} \cdot \delta \mathbf{r} = 0.$$

For many particles,

$$\sum_i \mathbf{R}_i \cdot \delta \mathbf{r}_i = 0.$$

SAY Constraint forces do not disappear because they are zero or unimportant. They disappear from the virtual-work equations because ideal constraint forces do no work on the allowed virtual displacements.

CHECK Emphasize the payoff: this is how coordinates adapted to the geometry can eliminate forces we do not want to solve for.

BOARD Applied-force virtual work:

$$\delta W = \sum_i \mathbf{F}_i \cdot \delta \mathbf{r}_i = \sum_j Q_j \delta q_j$$

with

$$Q_j = \sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}.$$

SAY Q_j is the generalized force conjugate to q_j . It need not have units of force.

BOARD Polar example:

$$Q_r = F_r, \quad Q_\theta = r F_\theta = \tau.$$

61–76 min: Newton projected onto allowed motion — d’Alembert

BOARD Newton for each particle:

$$m_i \mathbf{a}_i = \mathbf{F}_i + \mathbf{R}_i.$$

Move all terms to one side, dot with $\delta \mathbf{r}_i$, and sum:

$$\sum_i (\mathbf{F}_i + \mathbf{R}_i - m_i \mathbf{a}_i) \cdot \delta \mathbf{r}_i = 0.$$

For ideal constraints,

$$\sum_i (\mathbf{F}_i - m_i \mathbf{a}_i) \cdot \delta \mathbf{r}_i = 0.$$

SAY This is d’Alembert’s principle: Newton’s law projected onto the allowed virtual directions.

BOARD Insert

$$\begin{aligned} \delta \mathbf{r}_i &= \sum_j \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j : \\ 0 &= \sum_j \left[Q_j - \sum_i m_i \mathbf{a}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \right] \delta q_j. \end{aligned}$$

ASK What remains? *Rewrite the inertial projection in generalized-coordinate language.*

76–88 min: The kinetic-energy identity

BOARD Begin with

$$T = \frac{1}{2} \sum_i m_i \dot{\mathbf{r}}_i^2.$$

Because

$$\frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_j} = \frac{\partial \mathbf{r}_i}{\partial q_j},$$

we have

$$\frac{\partial T}{\partial \dot{q}_j} = \sum_i m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}.$$

BOARD Differentiate:

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} = \sum_i m_i \mathbf{a}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} + \sum_i m_i \dot{\mathbf{r}}_i \cdot \frac{d}{dt} \left(\frac{\partial \mathbf{r}_i}{\partial q_j} \right).$$

But

$$\frac{d}{dt} \left(\frac{\partial \mathbf{r}_i}{\partial q_j} \right) = \frac{\partial \dot{\mathbf{r}}_i}{\partial q_j},$$

so the second term is $\partial T / \partial q_j$. Therefore

$$\boxed{\sum_i m_i \mathbf{a}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j}.$$

CHECK This is the one algebraic identity worth deriving rather than quoting. It converts Newton's vector inertial term into generalized-coordinate form.

88–96 min: Lagrange equations and the return of $L = T - V$

BOARD Substitute the identity into d'Alembert:

$$0 = \sum_j \left[Q_j - \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} + \frac{\partial T}{\partial q_j} \right] \delta q_j.$$

Independent δq_j imply

$$\boxed{\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j.}$$

BOARD Conservative forces:

$$\delta W = -\delta V \quad \Rightarrow \quad \boxed{Q_j = -\frac{\partial V}{\partial q_j}.$$

Thus, with $L = T - V$,

$$\boxed{\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0.}$$

SAY Now we have reached Lagrange's equations from the Newton/virtual-work side. Lectures 4–5 reached them from stationary action. The two routes meet at the same structure.

OPTIONAL Nonconservative applied force:

$$\frac{d}{dt} L_{\dot{q}_j} - L_{q_j} = Q_j^{(\text{nc})}.$$

96–102 min: Running example — simple pendulum

BOARD

$$\mathbf{r}(\theta) = \begin{pmatrix} \ell \sin \theta \\ -\ell \cos \theta \end{pmatrix}, \quad T = \frac{1}{2} m \ell^2 \dot{\theta}^2, \\ V = mg\ell(1 - \cos \theta).$$

Therefore

$$L = \frac{1}{2} m \ell^2 \dot{\theta}^2 - mg\ell(1 - \cos \theta).$$

Euler–Lagrange:

$$\boxed{\ddot{\theta} + \frac{g}{\ell} \sin \theta = 0.}$$

ASK Where did the tension go?

SAY It did not cease to exist. The tension is radial, while the allowed virtual displacement is tangent, so its virtual work vanishes.

CHECK This is the payoff students should leave with: choose coordinates adapted to the constraint, and the unwanted reaction force need not appear in the reduced equation.

102–105 min: First symmetry clue — set up Lecture 7

BOARD Define generalized momentum:

$$p_j \equiv \frac{\partial L}{\partial \dot{q}_j}.$$

Euler–Lagrange says

$$\dot{p}_j = \frac{\partial L}{\partial q_j}.$$

If

$$\frac{\partial L}{\partial q_j} = 0,$$

then

$$p_j = \text{constant}.$$

SAY If a coordinate does not appear in the Lagrangian, its conjugate momentum is conserved. This is our first glimpse of symmetry and conservation.

NEXT Lecture 7: make that statement systematic — continuous symmetries and Noether’s theorem.

If behind: compress terminology and the mass-matrix glimpse; shorten the polar-force example; do *not* cut $d\mathbf{r}$ versus $\delta\mathbf{r}$, the d’Alembert projection, or the pendulum payoff.